ATIYAH-SINGER INDEX THEOREM FROM SUPERSYMMETRIC QUANTUM MECHANICS

HEATHER LEE

Master of Science in Mathematics
Nipissing University

2010
ATIYAH-SINGER INDEX THEOREM FROM
SUPERSYMMETRIC QUANTUM MECHANICS

HEATHER LEE

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF MASTER OF
SCIENCE IN MATHEMATICS

NIPISSING UNIVERSITY
SCHOOL OF GRADUATE STUDIES
NORTH BAY, ONTARIO

Heather Lee  August 2010
Declaration:

The undersigned hereby certify that they have read and recommend to the School of Graduate Studies the acceptance of the Major Research Paper entitled:

Supersymmetric Quantum Physical System and Altyah—Singer Index Theorem

by Heather Lee

of Master of Science, Mathematics

in partial fulfillment of the requirements for the degree
I hereby declare that I am the sole author of this Thesis or major Research Paper.

I authorize Nipissing University to lend this thesis or Major Research Paper to other institutions or individuals for the purpose of scholarly research.

I further authorize Nipissing University to reproduce this thesis or dissertation by photocopying or by other means, in total or in part, at the request of other institutions or individuals for the purpose of scholarly research.
Abstract

Some preliminary knowledge for understanding the Atiyah-Singer theorem, including differential manifolds, Cartan’s exterior algebra, de Rham cohomology, Hodge theory, Riemannian manifolds and Kähler manifolds, fibre bundle theory, and characteristic classes is reviewed. The Atiyah-Singer index theorems for four classical elliptic complexes, including the Gauss-Bonnet theorem of the de Rahm complex, Hirzebruch Signature theorem of the signature complex, Reimann-Roch theorems of the Dolbeault complex, and the index theorem for A-roof genus of spin complex, are introduced. Quantum mechanics is formulated from classical mechanics using the canonical and path integral quantizations. Supersymmetric quantum mechanics is introduced including the 0+1-dimensional supersymmetric nonlinear sigma model. By representing supersymmetric quantum theory as a graded Hilbert space, the identification between a supersymmetric quantum mechanical system and a certain elliptic complex is established. The Witten index is shown to be identified with the index of classical elliptic complexes. The Gauss-Bonnet theorem, Hirzebruch Signature theorem, the index theorem for A-roof genus, and the Reimann-Roch theorem are then derived in great detail using supersymmetric quantum mechanics. Finally, a mathematical conjecture is proposed about the mysterious cancelations between the terms in the proof of the Atiyah-Singer index theorem in the heat equation approach.
Acknowledgment

First and foremost, I would like to thank my supervisor Dr. Wenfeng Chen for the invaluable instruction and guidance he has provided. In addition, thanks to Dr. Vesko Valov and Dr. Alexandre Karassev for the knowledge they have passed on to me throughout the M.Sc. program. I would also like to acknowledge my second reader Dr. Murat Tuncali, and external examiner Dr. Achim Kempf. This paper is dedicated to my family and friends, for their continual and everlasting support.
Contents

1 Introduction and Background 5

2 Atiyah-Singer Index Theorem for Four Classical Elliptic Complexes 9
  2.1 Calculus on A Differential Manifold ........................................ 9
    2.1.1 Differential Manifold .................................................. 9
    2.1.2 Tangent Space, Cotangent Space and Differential Forms .......... 10
    2.1.3 Riemannian Manifold .................................................. 12
    2.1.4 Hodge Theorem and de Rham Cohomology ............................. 15
    2.1.5 Kähler Manifold ....................................................... 19
  2.2 Fibre Bundle Theory .......................................................... 22
    2.2.1 Topology of Fibre Bundle .............................................. 22
    2.2.2 Geometry of Vector Bundle .......................................... 25
  2.3 Characteristic Classes of Vector Bundles ............................... 27
    2.3.1 Chern Class ............................................................. 28
    2.3.2 Pontrjagin Class ....................................................... 30
    2.3.3 Euler Class ............................................................. 31
    2.3.4 Stiefel-Whitney Class ................................................ 32
  2.4 Classical Elliptic Complexes and Index Theorems ..................... 33
    2.4.1 Elliptic Complexes and Index of Elliptic Differential Operator . 33
    2.4.2 de Rham Complex and Gauss-Bonnet Theorem .......................... 36
    2.4.3 Signature Complex and Hirzebruch Signature Theorem ............ 38
    2.4.4 Dolbeault Complex and Riemman-Roch Theorem ..................... 39
    2.4.5 Spin Complex and Index Theorem for A-roof Genus ............... 39
B Grassmann Variable Calculus 83
C Clifford Algebra and Spinor 87
Chapter 1

Introduction and Background

The Atiyah-Singer index theorem, proved by M. Atiyah and I. Singer in 1963 [1, 2, 3, 4], is one of the most remarkable discoveries in modern mathematics. It states that the analytical index of an elliptic differential operator on a compact differential manifold is equal to the topological index. The analytical index counts the number of independent solutions of a partial differential equation obtained from the zero-eigenvalue equation of the elliptic differential operator. Meanwhile, the topological index characterizes the non-trivial topology of the vector bundles constructed on the compact manifold and reflects the global topological features of the differential manifold. Further, the topological index can be expressed as an integral of a certain characteristic class of vector bundles over the differential manifold. Therefore, this theorem brings together several distinct branches of modern mathematics including analysis, geometry and topology. Because of the Atiyah-Singer index theorem, the topological features of a differential manifold can be studied by means of mathematical analysis. Conversely, the existence, and even the number, of linearly independent solutions of a partial differential equation can be investigated by observing the global topology of a differential manifold on which the elliptic differential operator is defined.

The original proof by Atiyah and Singer relies on cobordism theory [1]. Later, Atiyah and Singer made use of K-theory to reprove the theorem and its numerous variants and generalizations [2, 3]. Both approaches are hard to understand. In 1973, Atiyah, Bott, and Patodi gave an excellent proof by using the heat equation [3, 5]. This method is based on the spectrum theory of a positive definite self-adjoint operator. The basic idea is to first use the inner product to define the adjoint operator of an elliptic differential operator. Then, construct two self-adjoint non-negative definite operators from the elliptic differential operator and its adjoint. The two self-adjoint positive-definite operators are usually called heat operators because their eigenvectors can furnish an orthonormal basis in which the solution to a heat equation, defined by the operators, admits an expansion. Despite two heat operators having paired non-zero eigenvalues, their eigenspaces corresponding to the zero eigenvalue can have different dimensions. Therefore, the index of the elliptic differential operator can be evaluated by calculating the difference between the traces of the exponentials of two heat operators [5]. This method has actually provided inspiration for theoretical physicists to use supersymmetric quantum mechanical systems to derive the Atiyah-Singer index theorem.

The idea of using supersymmetric quantum mechanics to prove the Atiyah-Singer index theorem was originally proposed by Witten when he investigated the dynamical supersymmetry breaking in various quantum field models possessing supersymmetry [6, 7]. In physics, quantum theory can be constructed from classical theory by a standard procedure called quantization. Classical mechanics has two equivalent descriptions: Lagrangian mechanics and Hamiltonian
mechanics. There are two equivalent quantization methods corresponding to the two classical formulations; that is, canonical quantization based on the Hamiltonian formulation, and path integral quantization based on the Lagrangian formulation. The mathematical description of a quantum theory constructed by means of canonical quantization is the theory of the operator and of the Hilbert space in functional analysis [11, 12]. A physical state is represented by a vector in the Hilbert space, while a physical quantity is represented by a self-adjoint (or Hermitian) operator. According to the statistical interpretation of quantum mechanics, once a basis of the Hilbert space, such as coordinate representation, is specified, the component of the state vector is relevant to the probability density for a physical phenomenon to take place. The measured value of a physical quantity is the average value of the operator representing it which is defined by the inner product of the Hilbert space. Further, the Hilbert space for a physical system has a special feature - it has at least one vector which can make the average value of the Hamiltonian operator minimal, termed in physics as the vacuum state vector.

A symmetry in a classical mechanical system is the invariance of the classical action under certain transformations of the variables in the Lagrangian. Classical action is defined as the integration of the Lagrangian over time. According to the celebrated Nöther theorem, the generator of a symmetry transformation must be a conservative physical quantity. At a quantum level, the conservative quantity is a time-independent operator. A classical theory possessing supersymmetry consists of two types of variables in either the Lagrangian or the Hamiltonian, the usual commutative variable and the anticommutative variable, called the Grassmann variable. In physics, the commutative and anticommutative variables are called *bosonic* and *fermionic* variables, respectively. Supersymmetry transformation is an exotic transformation that converts a bosonic variable into a fermionic variable and vice-versa. Therefore, the Hilbert space of a supersymmetric quantum theory is a graded space [13]. The generators of a supersymmetry transformation map the bosonic sector into the fermionic sector and vice-versa. An operator \((-1)^F\) can be defined to distinguish the bosonic sector from the fermionic sector. Because of the supersymmetry algebra, the dynamical breaking of supersymmetry depends on the existence of the vacuum state vectors that make the Hamiltonian take a zero eigenvalue. Hence, Witten defined the trace \(\text{Tr}(-1)^F\) to characterize the dynamical supersymmetry breaking, and realized it is actually identical to the index of an elliptic differential operator — the generator of supersymmetry transformations [7]. Subsequently, Alvarez-Gaumé [15, 16, 17], and Friedan and Windey [18] independently worked out detailed proofs. Further, Getzler gave a more mathematically rigorous description [19].

There are a number to advantages of using supersymmetry quantum mechanics to prove the Atiyah-Singer index theorem over other approaches. First, an explicit isomorphism between an elliptic complex and a supersymmetric quantum mechanical system can be established. Second, functional integration can be used to calculate the index straightforwardly. This evaluation method is much simpler and easier than the heat equation method [5]. In particular, it is believed that the supersymmetry can explain the mysterious cancelations between a large number of terms arising from the small parameter expansion used in the heat equation approach. The disadvantage of the supersymmetry approach is the lack of mathematical rigor in some aspects including the quantization procedure, operator ordering problem, and the mathematical definition of path integral etc.

The layout of this research paper is as follows: In Chapter 2, the Atiyah-Singer index theorem for four classical elliptic complexes and the relevant preliminary materials is reviewed, which covers calculus on differential manifolds (including Cartan’s exterior algebra, de Rham cohomology theory, Hodge decomposition theorem, Stokes’ theorem, Riemannian and Kähler manifolds), the global topology and local differential geometry of fibre bundle theory, characteristic classes
for vector bundles, and elliptic complexes. In Chapter 3, some necessary knowledge in quantum mechanics is introduced. This includes two equivalent descriptions of classical mechanics and the corresponding quantization methods of constructing quantum theory, supersymmetry algebra, and the structure of the Hilbert space of a supersymmetric mechanical system. Chapter 4 is the main part of the paper. It is shown in great detail how supersymmetric quantum mechanical systems can be identified with four classical elliptic complexes. Hence, four classical index theorems are derived using the functional integration method. Finally, in Chapter 5, a summary is given and a mathematical conjecture relevant to the proof of the Atiyah-Singer index theorem in the heat equation approach is proposed.
Introduction
Chapter 2

Atiyah-Singer Index Theorem for Four Classical Elliptic Complexes

2.1 Calculus on A Differential Manifold

2.1.1 Differential Manifold

A differential manifold is a generalization of the Euclidean space we are familiar with. Locally, an $n$-dimensional differential manifold looks like $\mathbb{R}^n$ or $\mathbb{C}^n$. By definition [8], an $n$-dimensional differential manifold $M$ is a Hausdorff topological space together with a family of pairs $\{(U_\alpha, \phi_\alpha)\}$ such that

- The $U_\alpha$’s are open sets covering $M$, i.e., $M = \bigcup_\alpha U_\alpha$.
- The $\phi_\alpha$’s are homeomorphisms such that $\phi_\alpha : U_\alpha \to V_\alpha$ where $V_\alpha$ is an open neighborhood in $\mathbb{R}^n$ or $\mathbb{C}^n$.
- Whenever $U_\alpha \cap U_\beta \neq \emptyset$, then the map $\phi_{\beta\alpha} \equiv \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is infinitely differentiable ($C^\infty$).

Because $\phi_\alpha$ maps each patch $U_\alpha$ on the manifold to $\mathbb{R}^n$ (or $\mathbb{C}^n$), a local coordinate system can thus be established on $U_\alpha$, and each point $P$ on the manifold can be represented by an ordered $n$-tuple of numbers $(x^1, x^2, \cdots, x^n)$. This last point is essential when defining a coordinate system on the manifold. $\phi_{\beta\alpha}$ relates two coordinate systems $\phi_\alpha$ and $\phi_\beta$ in the overlapping region $U_\alpha \cap U_\beta$. A pair $(U_\alpha, \phi_\alpha)$ is called a chart on $M$ and the family $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas [14]. Therefore, one can define local smooth functions on $M$ and use Calculus to study the geometry, and further the topology, of a differential manifold.
2.1.2 Tangent Space, Cotangent Space and Differential Forms

1. Tangent Space

Just like in calculus, the local shape of a plane curve can be studied by its tangent line at each point. The geometrical feature of a differential manifold \( M \) in the neighborhood of a point \( P \) can be studied using a tangent space \( T_P(M) \) — that is a vector space formed by tangent vectors of \( M \). Suppose a point \( P \in M \) is represented by the local coordinate \( x = (x^1, x^2, \ldots, x^n) \). Then one can define a smooth curve \( x(t) = (x^1(t), x^2(t), \ldots, x^n(t)) \) on \( M \) passing through \( P = x(0) \). We can consider a smooth function \( f(x) \) on \( M \). When \( f(x) \) is defined on the curve, it’s directional derivative along the curve at \( P \) is

\[
V_P f = \left. \frac{d}{dt} f(x(t)) \right|_{t=0} = \sum_{\mu=1}^{n} \frac{dx^\mu}{dt} \left. \frac{\partial f}{\partial x^\mu} \right|_P .
\]

From now on, in this paper we shall adopt Einstein’s convention that repeated indices are summed and the summation symbol will be omitted. The viewpoint in modern mathematics is that a tangent vector of \( M \) is just the differential operator appearing in (2.1),

\[
V = V^\mu(x) \frac{\partial}{\partial x^\mu}.
\]

A tangent space \( T_P(M) \) is the set of all the tangent vectors \( V \) at \( P \); hence it is a vector space with basis \( \{ \partial/\partial x^\mu \} \), \( \mu = 1, 2, \ldots, n \). Therefore, \( T_P(M) \) has the same dimension as \( M \).

2. Cotangent Space

According to functional analysis, there always exists a dual space associated with a vector space. The dual space of a vector space is defined as the set of linear functionals on the vector space. The cotangent space, \( T_P^*(M) \) at a point \( P \in M \) is the dual vector space to \( T_P(M) \). That is, if \( \omega \in T_P^*(M) \), then \( \omega : T_P(M) \to \mathbb{R} \). The basis vectors of \( T_P^*(M) \) are actually the differential line elements \( dx^\mu \), which are uniquely determined by the requirement

\[
dx^\mu \left( \frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu.
\]

Hence, a dual vector \( \omega \in T_P^*(M) \) can be written as

\[
\omega = \omega_\mu(x) dx^\mu.
\]

It is easy to see that \( V \) and \( \omega \) are actually contravariant and covariant vectors in the traditional theory of tensor analysis. Because of the \( x \)-dependence of the components \( V^\mu(x) \) and \( \omega_\mu(x) \), \( V \) and \( \omega \) are called contravariant and covariant vector fields, respectively.

Further, we can construct a larger tensor field space by taking the tensor product of \( k \) tangent spaces \( T_P(M) \) and \( l \) cotangent spaces \( T_P^*(M) \), whose elements \( T^k_l(M) \) are called type-(\( k,l \)) mixed tensor fields with \( l \) covariant and \( k \) contravariant indices,

\[
T^k_l \equiv T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} (x) \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_k}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_l}.
\]

3. Differential Form and Cartan’s Exterior Algebra
A completely antisymmetric covariant tensor field, that is, a tensor field which changes sign under the interchange of any two pairs of arguments, is called a differential form. A $p$-form at the point $P$ takes the following form,

$$\omega_p = \omega_{\mu_1, \cdots, \mu_p}(x) \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \quad (2.6)$$

In definition (2.6), $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ is the antisymmetrization of the tensor product $dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_p}$, ie.

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = \frac{1}{p!} \sum_{\sigma \in S_p} \text{sgn}(\sigma) \, dx^{\mu_{\sigma(1)}} \otimes \cdots \otimes dx^{\mu_{\sigma(p)}}, \quad (2.7)$$

where $S_p$ is a permutation group of $p$ elements, $\text{sgn}(\sigma) = 1$ for an even permutation, and $\text{sgn}(\sigma) = -1$ for an odd permutation. Obviously, there exists $0 \leq p \leq n$. When $p = 0$, the $0$-form is just a local function on $M$; when $p = n$, $dx^1 \wedge \cdots \wedge dx^n$ is the usual oriented volume element of $M$, which is usually called a top form. Let $\Lambda^p(x)$ denote the set of $p$-forms at $x$. Then $\Lambda^p(x)$ is a vector space of dimension $n!/[p!(n-p)!]$.

Further, define a $2^n$-dimensional graded linear space $\Lambda^*$ formed by the following direct sum,

$$\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^n. \quad (2.8)$$

Then we introduce a wedge product operation on $\Lambda^*$ as follows: given a $p$-form $\alpha_p \in \Lambda^p$, and a $q$-form $\beta_q \in \Lambda^q$, a $(p+q)$-form is created in the following way [14],

$$\alpha_p \wedge \beta_q = \text{antisymmetrizing} \quad \alpha_p \otimes \beta_q = (-1)^pq \beta_q \wedge \alpha_p. \quad (2.9)$$

Clearly $\Lambda^*$ is closed under the wedge product $\wedge$. Therefore, $\Lambda^*$ is a graded algebraic system under the operation of the wedge product $\wedge$, which is called Cartan’s exterior algebra.

4. Exterior Differentiation

Assume that all the differential forms in $\Lambda^*$ are smooth. Then there exists another well-defined operation on $\Lambda^*$, the exterior differential operator $d$ which is a mapping $d : \Lambda^p \rightarrow \Lambda^{p+1}$ defined by

$$d\omega_p = d(f_{\mu_1, \cdots, \mu_p} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) = \frac{\partial f_{\mu_1, \cdots, \mu_p}}{\partial x^\nu} \, dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \quad (2.10)$$

Note that the 1-form $dx^\nu$ is always inserted at the start of the wedge product. It can easily be proved that $d$ has the following properties:

1. $d$ is nilpotent,

$$d^2 = 0. \quad (2.11)$$

2. The rule for $d$ to differentiate the wedge product of a $p$-form $\alpha_p$ and a $q$-form $\beta_q$ is

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q. \quad (2.12)$$

5. Closed Form and Exact Form
Next, we introduce two special types of differential forms that will be used when discussing the de Rham cohomology.

A $p$-form $\alpha_p \in \Lambda^p$ is called a closed $p$-form if it satisfies

$$d\alpha_p = 0.$$  \hfill (2.13)

A $p$-form $\alpha_p \in \Lambda^p$ is called an exact $p$-form if it satisfies

$$\alpha_p = d\beta_{p-1},$$  \hfill (2.14)

for a $\beta_{p-1} \in \Lambda^{p-1}$. Clearly, because of Eq. (2.13), an exact form must be a closed form, while the reverse is not true.

2.1.3 Riemannian Manifold

1. Definition of Riemannian Manifold

Let $M$ be a smooth differential manifold. If there exists a non-degenerate positive-definite symmetric second-rank covariant tensor field $G$ at each point on $M$ where,

$$G = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu, \quad g_{\mu\nu} = g_{\nu\mu}, \quad \text{and} \quad \det [g_{\mu\nu}(x)] \neq 0,$$  \hfill (2.15)

then $M$ is called a Riemannian manifold. If it is not required that $(g_{\mu\nu})$ is positive-definite, $M$ is called a generalized Riemannian manifold. This type of Reimannian manifold was used by Einstein to describe gravitational theory. We discuss only Riemannian manifolds in this paper. $g_{\mu\nu}(x)$ is called a metric on $M$, from which the distance $ds$ between two infinitesimally nearby points, represented by the coordinates $(x^\mu)$ and $(x^\mu + dx^\mu)$, can be evaluated as,

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = ||dx^\mu||^2.$$  \hfill (2.16)

2. Vierbein (or Local Orthonormal Frame) on Riemannian Manifold

As a symmetric tensor, the metric tensor $g_{\mu\nu}(x)$ can be decomposed into vierbeins $e^a_\mu$, as [10],

$$g_{\mu\nu} = \delta_{ab}e^a_\mu(x)e^b_\nu(x), \quad \text{and} \quad \delta^{ab} = g^{\mu\nu}e^a_\mu(x)e^b_\nu(x).$$  \hfill (2.17)

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}(x)$,

$$g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho}.$$  \hfill (2.18)

g_{\mu\nu} can be decomposed into the following form,

$$g^{\mu\nu} = \delta^{ab}E^a_\mu E^b_\nu.$$  \hfill (2.19)

$E^a_\mu$ is the inverse of $e^a_\mu$ defined by

$$E^a_\mu = \delta_{ab}g^{\mu\nu}e^b_\nu,$$  \hfill (2.20)

which obeys

$$E^a_\mu e^b_\mu = \delta_a^b, \quad \text{and} \quad g_{\mu\nu}E^a_\mu E^b_\nu = \delta_{ab}.$$  \hfill (2.21)
In the above expressions, \( a = 1, 2, \cdots, n \) is the index of the local orthonormal frame.

The meanings of \( e^a_{\mu}(x) \) and \( E_a^\mu \) are as follows: \( e^a_{\mu}(x) \) is the orthogonal matrix which transforms the basis \( \{ dx^\mu \} \) of the cotangent space into an orthonormal basis \( \{ e^a \} \), while \( E_a^\mu \) transforms the basis \( \partial/\partial x^\mu \) of the tangent space \( T_x(M) \) into an orthonormal basis \( E_a \) where,

\[
e^a = e^a_{\mu}(x)dx^\mu, \quad \text{and} \quad E_a = E_a^\mu \frac{\partial}{\partial x^\mu}.
\]

Each \( e^a \) is called a \textit{vierbein one-form}.

3. Geometry of Riemannian Manifold

Riemannian geometry is usually studied by introducing a connection which can define parallel transportation of geometric objects on the manifold. Like a simple curve or a surface in \( \mathbb{R}^3 \), the geometry of a differential manifold is described by curvature and torsion tensor fields, both of which can be calculated using parallel transportation of geometric objects. Only a Riemannian manifold has the following two special features:

1. It is torsion-free.
2. The norm of a vector should be preserved in the parallel transport, which is usually called \textit{metricity}.

The connection satisfying these two conditions is called a \textit{Riemannian connection}.

There are two types of Riemannian connections on a Riemannian manifold: one is the Riemannian spin connection, and the other is the Levi-Civita connection (or called the \textit{Christoffel symbol} in physics). Both connections describe the same Riemannian geometry.

1. Riemannian geometry in terms of spin connection

We start from a more general \textit{affine} spin connection 1-form \( \omega^a_{\ b} = \omega^a_{\ b\mu}dx^\mu \). Then the parallel transportation of the orthonormal frames \( e^a \) on the manifold yield the torsion tensor. The parallel transportation of the \textit{affine} spin connection itself on the manifold gives the curvature tensor. Both of these are described by Cartan’s structure equation as:

\[
d e^a + \omega^a_{\ b} \wedge e^b = T^a = \frac{1}{2}T^a_{\ bc}e^b \wedge e^c,
\]

\[
d \omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b} = R^a_{\ bc} = \frac{1}{2}R^a_{\ bcd}e^c \wedge e^d.
\]

Then the Riemannian spin connection can be obtained by imposing two requirements listed above. From Eqs. (2.16), (2.17), (2.22) and (2.24), the metricity \( d (||dx^\mu||^2) = 0 \) leads to

\[
\omega_{ab} = -\omega_{ba},
\]

where \( \omega_{ab} = \delta^c_{\ [a} \omega^a_{\ b]} \). Further, from Eq. (2.23), the torsion-free requirement yields

\[
T^a = de^a + \omega^a_{\ b} \wedge e^b = 0.
\]

Once \( e^a \) is specified, \( \omega^a_{\ b} \) can be determined. Then the Riemannian curvature can be found by means of (2.24) and the geometry of the Riemannian manifold is fully determined.
2. Riemannian geometry in terms of Levi-Civita Connection

Expressed in terms of the Levi-Civita connection $\Gamma^\mu_{\rho\nu}$, the metricity and torsion-free requirements take the following forms [8],

Metricity: $D_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0$; (2.27)

Torsion-free: $T^\lambda_{\rho\mu} = \frac{1}{2} (\Gamma^\lambda_{\rho\mu} - \Gamma^\lambda_{\nu\mu}) = 0$. (2.28)

The Riemannian curvature tensor 2-form is defined by

$$R^\lambda_{\rho\mu\nu} = \frac{1}{2} \Gamma^\lambda_{\rho\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} \Gamma^\lambda_{\rho\mu\nu} dx^\mu \wedge dx^\nu.$$ (2.29)

where $\Gamma^\lambda_{\rho} = \Gamma^\lambda_{\nu\rho} dx^\nu$. Eqs. (2.27) and (2.28) can uniquely determine the Levi-Civita connection in terms of the metric,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left( \partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right).$$ (2.30)

Then the Riemannian curvature tensor can be calculated from (2.29) by

$$R^\lambda_{\rho\mu\nu} = \frac{1}{2} \omega_{a}^{\rho} e_{b}^{\mu} R_{b\mu\nu}.$$ (2.31)

$R_{\mu\nu\lambda\rho}$ has the following symmetric properties:

$$R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\rho\nu\lambda} = R_{\lambda\rho\mu\nu}, \quad \text{and} \quad R_{\mu\nu\lambda\rho} + R_{\lambda\mu\rho\nu} + R_{\nu\lambda\mu\rho} = 0.$$ (2.32)

3. Relation between Two Descriptions

The curvature 2-form $R^a_b$ can be decomposed as

$$R^a_b = \frac{1}{2} R^a_{bcd} e^c \wedge e^d = \frac{1}{2} R^a_{b\mu\nu} dx^\mu \wedge dx^\nu.$$ (2.33)

Then there is the relation

$$R^\lambda_{\rho\mu\nu} = E^\lambda_{\alpha} e^b_{\rho} R_{b\mu\nu}.$$ (2.34)

The torsion-free requirement (2.26) yields

$$\omega_{a\mu} \wedge e^b = \omega_{a\mu} \omega_{b\nu} dx^\mu \wedge dx^\nu = -d \left( e^b_{\nu} dx^\mu \right) = -\partial_\mu e^b_{\nu} dx^\mu \wedge dx^\nu$$

$$= - (\partial_\mu e^a_{\nu} - \Gamma^a_{\mu\nu} e^a_{\lambda}) dx^\mu \wedge dx^\nu \equiv - \left( D_\mu e^b_{\nu} \right) dx^\mu \wedge dx^\nu.$$ (2.35)

Hence we obtain

$$\omega_{b\mu} e^b_{\nu} = - \left( D_\mu e^b_{\nu} \right) = - (\partial_\mu e^a_{\nu} - \Gamma^a_{\mu\nu} e^a_{\lambda}),$$ (2.36)

which gives the relation between the Riemannian spin connection and the Levi-Civita connection,

$$\Gamma^\lambda_{\mu\nu} = E^\lambda_{\alpha} \left( \partial_\mu e^a_{\nu} + \omega_{b\mu} e^b_{\nu} \right).$$ (2.37)

These relations will be used in Sect. 4.4.
2.1.4 Hodge Theorem and de Rham Cohomology

Once we work on a Riemannian manifold $M$, the metric structure brings more operations and structures on $\Lambda^*(M)$.

1. Hodge $*$ operation

The Hodge $*$ operator is a map $*: \Lambda^p \rightarrow \Lambda^{n-p}$. That is, it converts $p$-forms into $(n-p)$-forms in $\Lambda^*$. On a Riemannian manifold it is defined by

$$ *(dx^\mu_1 \wedge \cdots \wedge dx^\mu_p) = \frac{1}{(n-p)!} g^{1/2} \varepsilon_{\mu_1 \cdots \mu_p} dx^{\nu_{p+1}} \wedge dx^{\nu_{p+2}} \wedge \cdots \wedge dx^{\nu_n}, $$

where $g = \det(g_{\mu\nu})$ and $\varepsilon_{\mu_1 \cdots \mu_n}$ is the $n$-dimensional totally antisymmetric tensor [8],

$$ \varepsilon_{\mu_1 \cdots \mu_n} = \begin{cases} 0, & \text{any two indices repeated;} \\ +1, & (\mu_1, \cdots, \mu_n) = \text{even permutation of } (1, \cdots, n); \\ -1, & (\mu_1, \cdots, \mu_n) = \text{odd permutation of } (1, \cdots, n). \end{cases} $$

It is a straightforward proof to show that the Hodge $*$ operator has the following property when acting on a $p$-form $\omega_p$,

$$ * * \omega_p = (-1)^{p(n-p)} \omega_p. $$

2. Inner Product defined with $*$

Using the Hodge operator we can define an inner product on $\Lambda^*$ as follows,

$$ (\alpha_p, \beta_p) = \int_M \alpha_p \wedge * \beta_p. $$

This inner product has the following properties:

$$ (\alpha_p, \beta_p) = (\beta_p, \alpha_p) = (*\alpha_p, *\beta_p), $$

$$ (\alpha_p, \alpha_p) \geq 0, \text{ and "=" stands only when } \alpha_p = 0. $$

3. Codifferential Operator Coclosed and Coexact Forms

The adjoint of the exterior differential operator $d$ can be defined by using the inner product (2.41). Define the adjoint of the exterior differential operator $d$, denoted by $\delta$, as,

$$ (\alpha_p, d\beta_{p-1}) = (\delta \alpha_p, \beta_{p-1}). $$

Clearly, $\delta$ is a mapping from $\Lambda^p$ to $\Lambda^{p-1},$

$$ \delta : \Lambda^p \rightarrow \Lambda^{p-1}. $$

It can be verified that when $\delta$ acts on a $p$-form,

$$ \delta = (-1)^{np+n+1} * d * . $$

It follows straightforwardly from either (2.43) or (2.45) that,

$$ \delta^2 = 0. $$

Recall that a $p$-form $\omega_p$ is called
• exact if \( \omega_p = d\alpha_{p-1} \), where \( \alpha_{p-1} \) is a \((p - 1)\)-form,
• closed if \( d\omega_p = 0 \).

Similar, a \( p \)-form \( \omega_p \) is called
• coexact if \( \omega_p = \delta\alpha_{p+1} \), where \( \alpha_{p+1} \) is a \((p + 1)\)-form,
• coclosed if \( \delta\omega_p = 0 \).

4. Laplacian Operator and Harmonic Form

The Laplacian operator can be defined from \( d \) and \( \delta \) as,
\[
\Delta = (d + \delta)^2 = d\delta + \delta d.
\] (2.47)

It takes the following form when acting on a \( p \)-form,
\[
\Delta\omega_p = d\delta\omega_p + \delta d\omega_p,
\] (2.48)

which is a map from \( \Lambda^p \) to itself. A \( p \)-form \( \alpha_p \) is called a harmonic form if it satisfies
\[
\Delta\alpha_p = 0.
\] (2.49)

The set of all harmonic \( p \)-forms is denoted by \( \text{Harm}^p(M) \).

It can be proved that \( \Delta \) has the following properties:

1. \( \Delta \) commutes with \( * \), \( d \) and \( \delta \),
\[
[\Delta, d] = [\Delta, \delta] = [\Delta, *] = 0.
\] (2.50)

2. \( \Delta \) is a self-adjoint positive definite operator. From the inner product (2.41), we have
\[
(\Delta\alpha_p, \beta_p) = (\alpha_p, \Delta\beta_p), \quad \text{and}
(\alpha_p, \Delta\alpha_p) = (\alpha_p, d\delta\alpha_p) + (\alpha_p, \delta d\alpha_p) = (\delta\alpha_p, \delta\alpha_p) + (d\alpha_p, d\alpha_p) \geq 0.
\] (2.51)

Further, Eq. (2.51) gives
\[
\Delta\alpha_p = 0 \text{ if and only if } d\alpha_p = 0 \text{ and } \delta\alpha_p = 0.
\] (2.52)

This means a harmonic form must be both closed and coclosed.

6. Stokes’ Theorem

Stokes’ Theorem is the generalization of the Fundamental Theorem of Calculus to a differential form. It states that given an \( n \)-dimensional, oriented manifold \( M \) with non-empty boundary and an \((n - 1)\)-form, \( \omega_{n-1} \), there exists
\[
\int_M d\omega_{n-1} = \int_{\partial M} \omega_{n-1}.
\] (2.53)

7. de Rham Cohomology
The de Rham cohomology tells us that closed differential forms can reflect topological information of a differential manifold through a duality established by Stokes’ theorem. Further, the equivalent classes of closed forms classified by exact forms constitutes a finitely generated Abelian group. Therefore, the de Rham cohomology is a link between differential topology and algebraic topology.

Let \( Z_p \) denote the set of all closed \( p \)-forms on \( M \), and let \( B_p \) denote the set of all exact \( p \)-forms on \( M \) so that:

\[
Z_p(M) = \{ \omega_p \in \Lambda^p : d\omega_p = 0 \} = \text{Ker} \, d_p,
\]

\[
B_p(M) = \{ \omega_p \in \Lambda^p : \omega_p = d\alpha_{p-1} \} = \text{Im} \, d_p.
\]

Because of \( d^2 = 0 \), we have

\[
B_p \subseteq Z_p.
\]

Therefore, all elements in \( Z_p \) can be classified by the equivalence relation defined as follows: for \( \alpha_p \in Z_p \) and \( \beta_p \in Z_p \),

\[
\alpha_p \sim \beta_p \text{ if and only if } \alpha_p - \beta_p = d\theta_{p-1} \in B_p.
\]

Consequently, all the elements \( \omega_p \) of \( Z_p \) can be classified into different equivalent classes, i.e., the cosets of \( B_p \),

\[
[\omega^1_p] = \omega^1_p + B_p, \ldots, \quad [\omega^k_p] = \omega^k_p + B_p, \ldots.
\]

The quotient space

\[
H_{DR}^p(M) = Z_p(M)/B_p(M) = \{ [\omega^1_p], \ldots, [\omega^k_p], \ldots, B_p \}
\]

is called the de Rham cohomology group. \( H_{DR}^p \) usually has finite order.

In the following we show that the dual of \( H_{DR}^p(M,R) \) is just the homology group \( H_p(M,R) \) from algebraic topology [9].

Let \( Z_p \) denote the set of \( p \)-chains with no boundary, which are called \( p \)-cycles, and let \( B_p \) denote the set of \( p \)-chains which are boundaries of \( p + 1 \)-chains:

\[
Z_p(M) = \{ C_p | \partial C_p = 0 \},
\]

\[
B_p(M) = \{ C_p | C_p = \partial C_{p+1} \}.
\]

Then \( B_p \subseteq Z_p \) because the boundary operator is nilpotent \( \partial^2 = 0 \). The simplicial homology group with real coefficients is the quotient group [9],

\[
H_p(M,R) = Z_p(M)/B_p(M).
\]

Let \( C_p \in Z_p \) and \( \omega_p \in Z_p \). Then the dual map \( \omega_p : C_p \rightarrow R \) is defined by

\[
\langle \omega_p, C_p \rangle = \int_{C_p} \omega_p.
\]

By using Stokes’ theorem, we have

\[
\int_{C_p} (\omega_p + d\alpha_{p-1}) = \int_{C_p} \omega_p + \int_{C_p} d\alpha_{p-1} = \int_{C_p} \omega_p + \int_{\partial C_p} \alpha_{p-1} = \int_{C_p} \omega_p,
\]

and

\[
\int_{C_p + \partial C_{p+1}} \omega_p = \int_{C_p} \omega_p + \int_{C_{p+1}} d\omega_p = \int_{C_p} \omega_p.
\]
Therefore, the above dual map can be expressed as,
\[ \langle [\omega_p], [C_p] \rangle = \int_{[C_p]} [\omega_p]. \] (2.64)

Consequently, the de Rham cohomology group and the simplicial homology group are naturally isomorphic,
\[ H^p_{\text{DR}}(M, R) \cong H_p(M, R). \] (2.65)

The $p$th Betti number of $M$ is defined as
\[ b_p = \dim H_p(M, R) = \dim H^p_{\text{DR}}(M, R), \] (2.66)
and the Euler characteristic is
\[ \chi(M) = \sum_{p=0}^{n} (-1)^p b_p. \]

8. Hodge Decomposition Theorem

**Hodge’s decomposition theorem** is a very useful result in the theory of partial differential equations (PDE). It provides a sufficient and necessary condition for the existence of the solution(s) to the differential equation $\Delta \omega_p = \eta_p$ in $\Lambda^p$.

**Hodge’s decomposition theorem**: Let $M$ be a compact Riemannian manifold without boundary. Then any $p$-form, $\omega_p \in \Lambda^p$ admits the following unique decomposition,
\[ \omega_p = \Delta \theta_p + \gamma_p, \] (2.67)
where $\gamma_p$ is a harmonic $p$-form, i.e., $\Delta \gamma_p = 0$.

From Eq. (2.43), we have
\[ (\Delta \theta_p, \gamma_p) = (\theta_p, \Delta \gamma_p) = 0. \]
So Hodge’s decomposition theorem can also be formally expressed as
\[ \Lambda^p(M) = \text{Harm}^p(M) \oplus \Delta \Lambda^p(M) = \text{Harm}^p(M) \oplus [\text{Harm}^p(M)]^\perp. \] (2.68)

Eq. (2.67) can be rewritten in the following form with Eq. (2.47),
\[ \omega_p = (d\delta + \delta d) \theta_p + \gamma_p = d\alpha_{p-1} + \delta \beta_{p+1} + \gamma_p. \] (2.69)
This shows that a $p$-form can be uniquely decomposed as a sum of exact, co-exact and harmonic forms. The Hodge decomposition theorem implies that the equation $\Delta \omega_p = \eta_p$ has a solution in $\Lambda^p(M)$ if and only if $\omega_p \in [\text{Harm}^p(M)]^\perp$.

9. Isomorphism between Harm$^p$(M) and $H^p(M, R)$

The Hodge decomposition theorem can lead to an important result. If $\omega_p \in H^p(M, R)$, then by Eqs. (2.52) and (2.69), we have
\[ d\omega_p = d\delta \beta_{p+1} = 0, \quad \text{and} \quad (\beta, d\delta \beta) = (\delta \beta, \delta \beta) = 0. \]
So $\delta \beta = 0$ and hence $\omega_p = d\alpha_{p-1} + \gamma_p$. Therefore, there exists
\[ \text{Harm}^p(M) \cong H^p(M, R) \cong H_p(M, R). \] (2.70)
2.1.5 Kähler Manifold

A Kähler manifold is a differential manifold that allows three compatible structures to be defined on it: a complex structure, a metric structure, and a symplectic structure. Hence, it is usually defined as a complex manifold with a Hermitian metric and a Kähler form.

1. Complex Manifold and Complex Tangent and Cotangent Spaces

An $n$-dimensional complex manifold is locally isomorphic to the complex vector space $\mathbb{C}^n$, i.e., there exist $\phi_i : U_i \subset M \rightarrow z = \{z^\alpha\}, \alpha = 1, \cdots, n$. Further, the local coordinate transformations defined by $\phi_{ji} \equiv \phi_j \circ \phi_i^{-1} : \phi_i (U_i \cap U_j) \rightarrow \phi_j (U_i \cap U_j)$ must be holomorphic, i.e., $z^{\beta} = f^{\beta}(z)$ depend only on $z$, not on its complex conjugate $\bar{z}$.

An $n$-dimensional complex manifold can be considered as a $2n$-dimensional real differential manifold with a complex structure. Hence, the complex coordinates can be represented by $2n$ real coordinates as $z^\alpha = x^\alpha + iy^\alpha$, where $y^\alpha \equiv x^{n+\alpha}$. The complex conjugate coordinates are defined in the usual way, $\bar{z}^\alpha = x^\alpha - iy^\alpha$.

Then we have

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2} \left( \frac{\partial}{\partial x^\alpha} - i \frac{\partial}{\partial y^\alpha} \right), \quad \frac{\partial}{\partial \bar{z}^\alpha} = \frac{1}{2} \left( \frac{\partial}{\partial x^\alpha} + i \frac{\partial}{\partial y^\alpha} \right);$$

$$dz^\alpha = dx^\alpha + idy^\alpha, \quad d\bar{z}^\alpha = dx^\alpha - idy^\alpha. \quad (2.71)$$

The complex tangent and cotangent spaces at each point of $M$ are defined in terms of their local bases [8]:

$$T_c(M) = \{\partial/\partial z^\alpha\}, \quad \bar{T}_c(M) = \{\partial/\partial \bar{z}^\alpha\},$$

$$\bar{T}_c(M) = \{dz^\alpha\}, \quad \bar{T}'_c(M) = \{d\bar{z}^\alpha\}. \quad (2.72)$$


A complex structure $J$ on a $2n$-dimensional real differential manifold is a smooth $(1, 1)$ type tensor field satisfying $J^2 = -I$, and it is a linear automorphism on the tangent space $T_P(M_{2n})$ at each point $P$, defined by

$$J \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x^{n+\alpha}} \equiv \frac{\partial}{\partial y^\alpha}, \quad J \frac{\partial}{\partial y^\alpha} = \frac{\partial}{\partial x^\alpha} = -\frac{\partial}{\partial y^\alpha}.$$

From the dual relation (2.3) between tangent and cotangent vectors, $dx^\mu (J \partial/\partial x^\nu) = (J dx^\mu)(\partial/\partial x^\nu)$, we obtain

$$J dx^\alpha = -dy^\alpha, \quad \text{and} \quad J dy^\alpha = -dx^\alpha.$$

Consequently, we have

$$J \frac{\partial}{\partial z^\alpha} = \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} + i \frac{\partial}{\partial x^\alpha} \right) = i \frac{\partial}{\partial \bar{z}^\alpha},$$

$$J \frac{\partial}{\partial \bar{z}^\alpha} = \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} - i \frac{\partial}{\partial x^\alpha} \right) = -i \frac{\partial}{\partial z^\alpha}. \quad (2.73)$$
and

\[
J \, dz^\alpha = -dy^\alpha + idx^\alpha = idz^\alpha, \\
J \, d\bar{z}^\alpha = -dy^\alpha - idx^\alpha = -id\bar{z}^\alpha.
\] (2.74)

Therefore, the vectors in \( T_c(M) \) and \( T^*_c(M) \) are called \([1,0]\) and \([0,1]\)-type vector fields, respectively, and those in \( T^*_c(M) \) and \( T^*_c(M) \) are called \([1,0]\) and \([0,1]\)-type one-forms. Subsequently, we have complex-valued \([p,q]\)-type smooth differential forms as,

\[
\omega_{p,q} = f_{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q}(z, \bar{z}) dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}.
\] (2.75)

All the \( \omega_{p,q} \) form a set \( \Lambda^{p,q} \), which is a module over a complex number ring.

The exterior differential operator on a sequence \( \{ \Lambda^{p,q} \} \) can be defined as,

\[
d = dx^\alpha \frac{\partial}{\partial x^\alpha} + dy^\alpha \frac{\partial}{\partial y^\alpha} = dz^\alpha \frac{\partial}{\partial z^\alpha} + d\bar{z}^\alpha \frac{\partial}{\partial \bar{z}^\alpha} \equiv \partial + \bar{\partial}.
\] (2.76)

Clearly, there exist

\[
\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}, \\
\bar{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1},
\] (2.77)

and

\[
\partial^2 = \bar{\partial}^2 = \partial \bar{\partial} + \bar{\partial} \partial = 0.
\] (2.78)

2. Hermitian Metric, Hodge \( * \) and Three Types of Laplacian Operators

A Hermitian metric is the counterpart of the Riemannian metric on a complex manifold. It can be expressed as the following form in terms of local coordinates \( \{ z^\alpha \} \),

\[
h = h_{\alpha\beta} dz^\alpha \otimes d\bar{z}^\beta,
\] (2.79)

where \( (h_{\alpha\beta}) \) is an \( n \times n \) non-degenerate positive-definite matrix, i.e., it is a Hermitian matrix where

\[
h = \det (h_{\alpha\beta}) > 0, \quad \text{and} \quad (h_{\alpha\beta})^T = (h_{\beta\alpha}).
\] (2.80)

Further, the real part of \( h \) gives the usual Riemannian metric, while its imaginary part is a \([1,1]\)-type differential form i.e., a simplectic structure in the context of a real \( 2n \)-dimensional manifold,

\[
ds^2 = \frac{1}{2} h_{\alpha\beta} \left( dz^\alpha \otimes d\bar{z}^\beta + d\bar{z}^\beta \otimes dz^\alpha \right) = h_{\alpha\beta} dz^\alpha d\bar{z}^\beta,
\]

\[
K = \frac{i}{2} h_{\alpha\beta} \left( dz^\alpha \otimes d\bar{z}^\beta - d\bar{z}^\beta \otimes dz^\alpha \right) = h_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta = K.
\] (2.81)

A complex manifold with a Hermitian metric is called a Hermitian manifold.
Subsequently, in the same way as on a Riemannian manifold, the complex Hodge $\ast$ operator can be defined on a $[p, q]$-form as,

$$\ast \left( dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q} \right)$$

$$\equiv \frac{1}{(n-p)! (n-q)!} \epsilon^{\alpha_1_\cdots_\alpha_p_\beta_1_\cdots_\beta_q} \epsilon_{\gamma_1_\cdots_\gamma_p_\delta_1_\cdots_\delta_q}$$

$$\times \epsilon_{\gamma_1_\cdots_\gamma_p_\delta_1_\cdots_\delta_q} dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_{n-p}} \wedge d\bar{z}^{\sigma_1} \wedge \cdots \wedge d\bar{z}^{\sigma_{n-p}}.$$  \hspace{1cm} (2.82)

The inner product between two $[p, q]$-forms is defined as,

$$(\alpha_{p,q}, \beta_{p,q}) = \int \alpha_{p,q} \wedge \ast \bar{\beta}_{p,q}. \hspace{1cm} (2.83)$$

Using the inner product, we can introduce the adjoint operator of $d$, the co-differential operator $\delta$, which is defined by $(d\alpha_{p,q}, \beta_{p+1,q+1}) = (\alpha_{p,q}, \delta \beta_{p+1,q+1})$ and found to be the following form,

$$\delta = -\ast d \ast = -\ast (\partial + \bar{\partial}) \ast = \partial^* + \bar{\partial}^*,$$

$$\delta : \Lambda^{p,q} \rightarrow \Lambda^{p-1,q-1}, \hspace{1cm} \partial^2 = 0.$$  \hspace{1cm} (2.84)

where

$$\partial^* = -\ast \partial : \Lambda^{p,q} \rightarrow \Lambda^{p-1,q};$$

$$\bar{\partial}^* = -\ast \bar{\partial} : \Lambda^{p,q} \rightarrow \Lambda^{p,q-1}.$$  \hspace{1cm} (2.85)

Clearly there exist

$$\partial^* \square = 0, \hspace{1cm} (\partial \alpha_{p,q}, \beta_{p+1,q}) = (\alpha_{p,q}, \partial^* \beta_{p+1,q});$$

$$\bar{\partial}^* \square = 0, \hspace{1cm} (\bar{\partial} \alpha_{p,q}, \beta_{p+1,q+1}) = (\alpha_{p,q}, \bar{\partial}^* \beta_{p+1,q+1}).$$  \hspace{1cm} (2.86)

Then there are three Laplacian operators that can be defined as,

$$\Delta = (d + \partial^2) = d\delta + \delta d,$$

$$\Delta' = 2(\partial + \partial^*) \square = 2(\partial \partial^* + \partial^* \partial)$$

$$\Delta'' = 2(\bar{\partial} + \bar{\partial}^*) \square = 2(\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}).$$  \hspace{1cm} (2.87)

All of them are self-adjoint operators such that,

$$(O \alpha_{p,q}, \beta_{p,q}) = (\alpha_{p,q}, O \beta_{p,q}), \hspace{1cm} O = \Delta, \Delta', \Delta''.$$  \hspace{1cm}

Further, for a compact Hermite manifold without boundary, there exists a generalized Hodge decomposition theorem,

$$\omega_{p,q} = \partial \alpha_{p-1,q} + \partial^* \beta_{p+1,q} + \gamma_{p,q}, \hspace{1cm} \Delta' \gamma_{p,q} = 0.$$  \hspace{1cm}

3. Geometry of Kähler Manifold

When the differential form $K$ defined in Eq. (2.81) is closed, ie.

$$dK = 0,$$  \hspace{1cm} (2.88)
then the Hermitian manifold becomes a Kähler manifold and $K$ is called a Kähler form. It can be proved that on a Kähler manifold, the three Laplacian operators listed in (2.87) are identical, $\Delta = \Delta' = \Delta''$.

Similarly to the Riemannian manifold, a Kähler manifold has vanishing torsion. The non-vanishing connections are only

$$
\Gamma_{\beta \gamma}^\alpha = h^{\alpha \delta} \frac{\partial h_{\beta \delta}}{\partial z^\gamma}, \quad \text{and} \quad \Gamma_{\beta \gamma}^\alpha = h^{\alpha \delta} \frac{\partial h_{\beta \delta}}{\partial z^\gamma}.
$$

The curvature is a $[1,1]$-type form [8]

$$
R_{\beta \gamma}^\alpha = R_{\beta \gamma \delta} d z^\alpha \wedge d z^\delta, \quad R_{\beta}^\alpha = R_{\beta}^\alpha,
$$

and the non-vanishing components of the curvature tensor fields are only

$$
R_{\beta \gamma \delta}^\alpha = -\frac{\partial \Gamma_{\beta \gamma}^\delta}{\partial z^\delta}, \quad \text{and} \quad R_{\beta \gamma \delta}^\alpha = -R_{\beta \gamma \delta}^\alpha.
$$

2.2 Fibre Bundle Theory

The theory of fibre bundles studies differential geometry and topology of differential manifolds by observing a twisted topological product of two differential manifolds. It is a powerful mathematical tool using local differential geometry data to study global topology of a differential manifold.

2.2.1 Topology of Fibre Bundle

1. Definition of Fibre Bundle

A (differentiable) fibre bundle is a mathematical structure made up of six parts: $(E, M, F, \pi, \phi, G)$, each of which can be explained as follows:

- $M$ is an $n$-dimensional manifold called a base. It is assumed that $M$ is covered by a set of open coordinate neighborhoods $U_\alpha$;
- $F$ is called a fibre, which is a $k$-dimensional manifold and is located at each point of $M$. The dimension of a fibre bundle usually means the dimension of the fibre $F$;
- The topological space $E$ is called the total space which is locally like $U_\alpha \times F$ in each neighborhood $U_\alpha \subseteq M$;
- $\pi$ is a surjective projection map such that $\pi : E \to M$, which shrinks each fibre to a point of $M$;
- $\phi = \{ \phi_\alpha \}$ is a set of homeomorphisms such that $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. $\phi_\alpha$ is called a local trivialization. Because of $\phi_\alpha$, a local coordinate $(x; z) \equiv (x^\mu_1 \cdots x^\mu_n; z^i_1 \cdots z^i_k)$ can be assigned to each point of $E$. Further, $\phi_\alpha$ satisfies $\pi \phi_\alpha^{-1}(x; z) = x$. 

• In the overlap \( U_\alpha \cap U_\beta \), \( \phi_\alpha \circ \phi_\beta^{-1} \) is an invertible continuous map of the form

\[
\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \longrightarrow (U_\alpha \cap U_\beta) \times F.
\]

For a fixed \( x \in U_\alpha \cap U_\beta \), \( \phi_{\alpha\beta}(x) = \phi_\alpha \circ \phi_\beta^{-1}(x) \) are transformation matrices in a fibre space \( F_x \) located at \( x \), which are called\,\,transition functions.\,\,Clearly, the transition functions satisfy

\[
\begin{align*}
\phi_{\alpha\alpha}(x) &= I_{k \times k}, \\
\phi_{\alpha\beta}(x) &= [\phi_{\beta\alpha}(x)]^{-1} \quad \text{for} \quad x \in U_\alpha \cap U_\beta, \\
\phi_{\alpha\gamma} \circ \phi_{\beta\gamma}(x) &= \phi_{\alpha\gamma}(x) \quad \text{for} \quad x \in U_\alpha \cap U_\beta \cap U_\gamma.
\end{align*}
\]

Therefore, the transition functions form a group \( G \equiv \{\phi_{\alpha\beta}(x)\} \), which is a transformation group on the fibre space \( F_x \). \( G \) is usually called the structure group of the fibre bundle \( E \).

When \( E = M \times F \) and the transition function is an identity, the bundle \( E \) is called a trivial fibre bundle.

Another important notion in fibre bundle theory is the cross section, or simply the section, of a fibre bundle. A section of a fibre bundle is a one-to-one smooth map \( S : U \subset M \longrightarrow \pi^{-1}(U) \), which assigns a function \( S(x) \in F_x \) for every point \( x \in M \). Note that a local section is defined only in a neighborhood \( U \) of \( M \) [14]. Only a trivial bundle can have a well-defined global section.

A simple example of a non-trivial fibre bundle is the M"{o}bius strip. The M"{o}bius strip is made up of the base manifold \( S^1 \) together with the fibre \( F \) being a line segment \( I = [-1, 1] \). \( S^1 \) is covered by two coverings, namely two arcs which are greater than the semicircle with structure group \( G = \{e, g\} \).  

2. Some Typical Fibre Bundles

The following is a list of some typical fibre bundles:

• An \( k \)-dimensional real vector bundle \( E \) is a fibre bundle whose fibre \( F \) is a real vector space of dimension \( k \). The structure group is \( GL(k, \mathbb{R}) \). If a Riemannian metric is defined on \( F \), then \( GL(k, \mathbb{R}) \) reduces to its subgroup \( O(k) \), the orthogonal group. Further, if \( F \) is oriented, then \( O(k) \) reduces to its subgroup \( SO(k) \), the special orthogonal group [8]. Let \( (e_1, e_2, \ldots, e_k) \) be a basis of \( F \). Then the local section \( S(x) \) at a point \( x \in M \) is a function on \( M \), and also a vector in \( F \)

\[
S(x) = \sum_{i=1}^{k} e_i(x) z^i(x).
\]  \hspace{1cm} (2.92)

For example, a tangent bundle \( T(M) \), whose fibres are tangent spaces at each point of a differential manifold \( M \), is a typical vector bundle. So is the cotangent bundle \( T^*(M) \).

• A complex vector bundle is similar to a real vector bundle but has the fibre \( F \) with the structure of \( \mathbb{C}^k \), and transition functions belonging to \( GL(k, \mathbb{C}) \). When an inner product is defined on \( F \), \( GL(k, \mathbb{C}) \) reduces to its subgroup \( U(k) \), the unitary group. Further, if \( F \) is oriented, then \( U(k) \) reduces to its subgroup \( SU(k) \), the special unitary group.

• A spin bundle is a bundle whose fibre is a space of spinors (see Appendix C), and the structure group is the spin group \( \text{Spin}(k) \). \( \text{Spin}(k) \) is a covering group of \( SO(k) \). For example, \( \text{Spin}(3) = \text{SU}(2) \) is a double covering of \( SO(3) \).
• A principle bundle, denoted $P$, is made up of a fibre that is a Lie group $G$ with transition functions belonging to $G$ [8].

We will focus on vector bundles and spin bundles for later use.

3. Dual Bundle, Direct Sum and Tensor Product of Vector Bundles

New bundles can be constructed from some existing fiber bundles. For later use, we introduce only the dual bundle of a vector bundle, the line bundle, the direct sum and tensor product of vector bundles.

• Dual Bundle

Let $E$ be a vector bundle whose fiber at each point $x \in M$ is a $k$-dimensional vector space $V$ with transition functions of $k \times k$ matrices $\Phi(x)$. Then the dual bundle $E^*$ is a vector bundle whose fibre is the dual space $V^*$ with transition functions of $k \times k$ matrices $[\Phi(x)]^{-1}$.

• Line Bundle

A line bundle is a vector bundle whose fibre is a one-dimensional vector space. It is a family of real lines $R$ or complex number $C$ located at each point of the base manifold $M$. The structure group of a line bundle is $GL(1,R)$ if it is real or $GL(1,C)$ if it is complex. Both $GL(1,R)$ and $GL(1,C)$ are Abelian groups.

• Direct Sum of Vector Bundles

Let $A$ and $B$ be two vector bundles over a differential manifold $M$. Suppose that the fibre of $A$ is a $j$-dimensional vector space $V$ with basis $\{e_1, \cdots, e_j\}$ and the transition functions are $j \times j$ matrices $X$, while the fibre of $B$ is a $k$-dimensional vector space $W$ with basis $\{f_1, \cdots, f_k\}$ and the transition functions composed of $k \times k$ matrices $Y$. Then the direct sum $A \oplus B$ can be obtained by taking both the direct sum of $V$ and $W$ at each point $x \in M$, along with the direct sum of $X$ and $Y$ where,

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix},$$

$$V \oplus W = \text{span} \{e_1, \cdots, e_j; f_1, \cdots, f_k\}.$$  \hspace{1cm} (2.93)

Thus the direct sum $A \oplus B$ is a $j + k$-dimensional vector bundle with fibre $V \oplus W$, whose transition functions are $(j + k) \times (j + k)$ matrices.

• Tensor Product of Vector Bundles

Taking the tensor product of $V$ and $W$ and the direct product of $X$ and $Y$ at each point $x \in M$ as,

$$V \otimes W = \text{span} \{e_p \otimes f_q : p = 1, \cdots, j; q = 1, \cdots, k\},$$

$$X \times Y = Y \times X = (X_{pq}Y) = (Y_{qq}X),$$  \hspace{1cm} (2.94)

we obtain a tensor product $A \otimes B$, which is a tensor bundle of $jk$ dimensions with fibers being $V \otimes W$ and transition functions being $jk \times jk$ matrices. The generalization to the tensor product of many bundles is straightforward.
The local differential geometry of a vector bundle is characterized by its curvature, which can be detected by parallel transport of the section of the fibre bundle along a path in the base manifold. Actually, curvature measures how much a section has changed when it has been parallelly transported along a path. This parallel transportation of the section is defined by the connection of the bundle. This is because the section of a fibre bundle, e.g., a vector field in a vector bundle, is an extension of a function defined on the base manifold. One can directly calculate the differentiation of a smooth scalar function, whereas to differentiate a section one must define how to compare the sections of a fibre located at one point with the section of a fibre at a point nearby. This comparison requires a connection on the bundle.

Let $X$ be a tangent vector at a point $x \in M$ and let $S(x)$ be a section of a vector bundle $E$. The connection, denoted by $\nabla$, is a rule that defines the directional derivative $\nabla_X S(x)$ and compares it to another section of $E$ at a neighborhood point along the curve whose tangent vector is $X$. According to the definition of the directional derivative along a vector, we have

$$\nabla_X S(x) = X(\nabla S(x)) = \langle X, \nabla S(x) \rangle. \quad (2.95)$$

Let $\Gamma(E)$ denote the set of sections of $E$. Then from (2.95), $\nabla$ should be a map defined as,

$$\nabla : \Gamma(E) \longrightarrow \Gamma[E \otimes T^*(M)],$$

$$S(x) \longrightarrow \nabla S(x),$$

$$\nabla S(x) = \nabla \left[ e_i(x) z^i(x) \right] = e_i(x) \otimes dz^i(x) + [\nabla e_i(x)] \otimes z^i \equiv e_i(x) \otimes dz^i(x) + e_j(x) \omega^i_j(x) \otimes z^i, \quad (2.96)$$

where the action of $\nabla$ on the basis $\{e_i\}$ is defined by

$$\nabla e_i(x) = e_j(x) \omega^i_j(x), \quad (2.97)$$

and

$$\omega^j_i(x) = \omega^j_{\mu i} dx^\mu \quad (2.98)$$

is the one-form connection of the vector bundle $E$.

With the above connection, a parallel transportation can be defined as follows: Let $x(t) = (x^1(t), x^2(t), \cdots, x^n(t))$ be a curve in $M$. Then the tangent vector to $x(t)$ is $\dot{x}(t) = dx(t)/dt$. When $S(x)$ takes a parallel transportation along $x(t)$, the component of $S(x)$ in the direction of the tangent vector $\dot{x}(t)$ should remain constant. That is, the change $\nabla S(x)$ should be orthogonal to $\dot{x}(t)$. Hence, the directional derivative of $S(x)$ in the direction of the tangent vector is zero,

$$\nabla_{\dot{x}} S(x) = \langle \dot{x}, \nabla S(x) \rangle = \nabla_{d/dt} \left[ e_i(x) z^i(x) \right] = 0.$$

This leads to [8]

$$\left( \partial_\mu z^j + \omega^j_{\mu i} z^i \right) \dot{x}^\mu e_j = 0.$$

As in Eq. (2.35), we denote

$$D_\mu \equiv \partial_\mu + \omega_\mu, \quad (2.99)$$

which is called a covariant derivative.
The curvature then arises as a matrix-valued 2-form, defined as
\[ \Omega^i_{\ j} = d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j} = (D\omega)^i_{\ j} = \frac{1}{2} \Omega^i_{\ j\mu\nu} dx^\mu \wedge dx^\nu. \] (2.100)

From (2.98), the components of the curvature tensor field are
\[ \Omega^i_{\ j\mu\nu} = \partial_\mu \omega^i_{\ j\nu} - \partial_\nu \omega^i_{\ j\mu} + [\omega^\mu_{\ j}, \omega^\nu_{\ i}]_{\ j} = (D\omega^\nu_{\ \mu} - D\omega^\mu_{\ \nu})^i_{\ j}. \] (2.101)

Applying exterior differentiation to the matrix form of the curvature (2.100),
\[ d\Omega = d\omega \wedge \omega \wedge d\omega = (\Omega - \omega \wedge \omega) - \omega \wedge (\Omega - \omega \wedge \omega) \]
\[ = \Omega \wedge \omega \wedge \Omega, \] (2.102)
we obtain the Bianchi identity
\[ D\Omega = d\omega + \omega \wedge \Omega - \Omega \wedge \omega = 0, \] (2.103)
which plays an essential role in constructing characteristic classes of fibre bundles.

If the vector bundle is a tangent or cotangent bundle, torsion also plays a role in the local geometry. However, since we consider only the case where the base manifold \( M \) is a Riemannian manifold, the Levi-Civita connection is used to define the parallel transportation and so the torsion tensor vanishes.

The following shows the transformations of connection and curvature under the action of transition functions. Let \( g(x) \equiv \phi_{\alpha\beta}(x) \) be transition functions on \( U_\alpha \cap U_\beta \). Then the transformations of the basis \( \{e_i(x)\} \) on a fibre \( F \) at \( x \in U_\alpha \cap U_\beta \), and on the coordinate components \( \{z^i(x)\} \), are
\[ e'_i(x) = e_i(x)g^{-1}_{ij}(x), \quad \text{and} \quad z'^h(x) = g_{ij}(x)z^j(x), \] (2.104)
so that the section \( S(x) \) is invariant, ie.
\[ S(x) = e_i z^i = e'_i z'^h(x) = S'(x). \]
In (2.104), \( g_{ij}(x) \) \( (i, j = 1, \ldots, k) \) are matrix elements of \( g \).

Requiring that \( \nabla (S(x)) \) is invariant under the basis change (2.104), and using (2.97) where,
\[ \nabla S(x) = \nabla' S'(x) \]
\[ = e_i(x) \otimes dz^i(x) + e_j(x)\omega^i_{\ j}(x) \otimes z^i \]
\[ = e'_i(x) \otimes dz'^i(x) + e'_j(x)\omega'^i_{\ j}(x) \otimes z'^i, \] (2.105)
one can find the connections (2.98) and (2.100) transform as follows:
\[ \omega'^i_{\ j} = g_{ik}\omega^k_{\ ij}g^{-1}_{lj} + g_{ik}dg_{kj}, \quad \Omega'^i_{\ j} = g_{ik}\Omega^k_{\ ij}g^{-1}_{lj}. \] (2.106)
In matrix form, the above transformations can be expressed as
\[ \omega = g\omega g^{-1} + gdg^{-1}, \quad \text{and} \quad \Omega = g\Omega g^{-1}. \] (2.107)

Finally, we briefly mention the parallel transportation of the section of a spin bundle based on a Riemannian manifold. The section \( \psi(x) \) of a spin bundle is a spinor function with \( 2^{n/2} \) components (see Appendix C): \( \psi(x) = (\psi_\alpha), \alpha = 1, \ldots, 2^{n/2} \). The parallel transportation of
Characteristic Classes

ψ(x) ω can only be done by a spin connection, which is a 1-form on the base manifold but takes the value in the Lie algebra of a spin group,

$$\omega^\alpha \beta = (\omega_\mu)^\alpha \beta dx^\mu = \frac{i}{2} \omega_{\mu ab} (\gamma_\alpha \gamma_b) \alpha \beta dx^\mu,$$

$$\gamma^a = e^a_\mu \gamma^\mu, \quad a, \mu = 1, \cdots, n, \quad \alpha, \beta = 1, \cdots, 2^{[n/2]}.$$ (2.108)

Note that $i/4 [\gamma_a, \gamma_b]$ is the spinor representations for the generators of the spin group Spin($n$).

The covariant derivative on the section defined by the spin connection is

$$D_\mu \psi_\alpha(x) = \partial_\mu \psi_\alpha(x) + \omega^\alpha_\mu \psi_\beta(x).$$ (2.109)

We use frequently the positive definite Dirac operator,

$$D = \gamma^a E^\mu_a D_\mu = \gamma^a E^\mu_a \left( \frac{\partial}{\partial x^\mu} + \frac{i}{4} \omega_{\mu ab} [\gamma_a, \gamma_b] \right),$$ (2.110)

where $E^\mu_a$ is given in (2.20), and $\gamma^a, a = 1, 2, \cdots, n$ are $2^{[n/2]} \times 2^{[n/2]} \gamma$-matrices introduced in Appendix C.

2.3 Characteristic Classes of Vector Bundles

In general, a characteristic class is a closed differential form on the base manifold of a fibre bundle. It is usually made up of a symmetric polynomial of the curvature which is invariant under the transformation (2.107). For a non-trivial fibre bundle, the characteristic class cannot be written as an exact form and so its integration over a compact base manifold with no boundary yields a non-zero topological number. Therefore, a characteristic class is a cohomology class of the base manifold. It provides a way of using topological invariants to distinguish between inequivalent fibre bundles [8] and to measure the deviation of a fibre bundle from the corresponding trivial bundle.

Let $P(\Omega)$ be a symmetric polynomial of the matrix-valued curvature 2-form $\Omega$ satisfying the following invariance under the transformation (2.107),

$$P(g^{-1} \Omega g) = P(\Omega)$$ (2.111)

for any $g \in G$, where $G$ is the structure group of a real or complex vector bundle. Without a loss of generality, we can simply choose [17]

$$P_m(\Omega) = \text{Tr} \left( \Omega \wedge \Omega \wedge \cdots \wedge \Omega \right)_m = \text{Tr} (\Omega^m)$$ (2.112)

to illustrate the features of a symmetric invariant polynomial. $P(\Omega)$ has the following properties:

1. $P_m(\Omega)$ is a closed $2m$-form on $M$:

$$dP_m(\Omega) = m\text{Tr} \left( d\Omega \Omega^{m-1} \right) = m\text{Tr} \left[ (d\omega + \omega \Omega - \Omega \omega) \Omega^{m-1} \right] = 0,$$ (2.113)

where we have used (2.12), the Bianchi identity (2.103), and the following cyclic property for matrix-valued $p_1$-differential forms,

$$\text{Tr} \left( \alpha p_1 \wedge \cdots \wedge \alpha p_{k-1} \wedge \alpha p_k \right) = (-1)^{p_1 + p_2 + \cdots + p_k} \text{Tr} \left( \alpha p_k \wedge \alpha p_1 \wedge \cdots \wedge \alpha p_{k-1} \right).$$
2. The integration of $P_m(\Omega)$ over the base manifold $M_{2m}$ is independent of the choice of the connection $\omega$ and is a topological invariant. This fact can be shown as follows [8, 17]:

Choose two arbitrary connections $\omega_0$ and $\omega_1$, and define the interpolation

$$\omega_t = \omega_0 + (\omega_1 - \omega_0) t, \quad t \in [0, 1],$$

$$\Omega_t = d\omega_t + \omega_t \wedge \omega_t.$$ 

One can find

$$\frac{d}{dt} P_m(\Omega_t) = m d \left( \text{Tr} \left[ (\omega_1 - \omega_0) \wedge \Omega_t^{m-1} \right] \right).$$

Then

$$\int_0^1 dt \frac{d}{dt} P_m(\Omega_t) = P_m(\Omega_1) - P_m(\Omega_0) = m d \left( \int_0^1 dt \text{Tr} \left[ (\omega_1 - \omega_0) \wedge \Omega_t^{m-1} \right] \right).$$

Hence by Stokes' theorem, on a $2m$-dimensional compact base manifold $M$ with no boundary, we have

$$\int_M P_m(\Omega_1) = \int_M P_m(\Omega_0). \quad (2.114)$$

### 2.3.1 Chern Class

#### 1. Chern Form

The Chern classes are the characteristic classes defined on a $k$-dimensional complex vector bundle $E$ with transition functions from $GL(k, C)$. The curvature 2-form $\Omega$ takes its value from the Lie algebra of $GL(k, C)$. The total Chern form is defined as

$$c(E) \equiv \det \left( I + \frac{i}{2\pi} \Omega \right) = 1 + c_1(\Omega) + c_2(\Omega) + \ldots \quad (2.115)$$

From this definition, each Chern form is a symmetric invariant polynomial in $\Omega$ and is composed of the basic polynomial $P_m(\Omega)$ defined in (2.112). For example,

$$c_0 = 1, \quad c_1 = \frac{i}{2\pi} \text{Tr} \Omega, \quad c_2 = \frac{1}{8\pi^2} \left[ \text{Tr} \left( \Omega^2 \right) - (\text{Tr} \Omega)^2 \right], \quad \cdots,$$

where $c(\Omega)$ is always a finite sum since $c_j = 0$ when $2j > n = \dim M$. From Eqs. (2.113) and (2.114), each Chern form $c_j(\Omega)$ defines the $2j^{th}$ cohomology class of $M$,

$$c_j(\Omega) \in H^{2j}(M), \quad (2.116)$$

and the integration on a $2j$-cycle $\gamma_{2j}$ in $M$ with integer coefficients gives an integer, the Chern number, regardless of the connection used.

From the definitions of dual vector bundle $E^*$ and the direct sum $A \oplus B$ of vector bundles $A$ and $B$, one can easily find [8]

$$C(E^*) = 1 - c_1(\Omega) + c_2(\Omega) - \cdots \quad (2.117)$$

and

$$C(A \oplus B) = C(A) \wedge C(B). \quad (2.118)$$
2. Chern Character

The Chern character is also an invariant polynomial defined by

\[
\text{ch}(E) = \text{Tr} \left[ \exp \left( \frac{i}{2\pi} \Omega \right) \right] = \sum_j \frac{1}{j!} \text{Tr} \left( \frac{i}{2\pi} \Omega \right)^j = k + c_1(\Omega) + \frac{1}{2} \left[ c_1(\Omega)^2 - 2c_2(\Omega) \right] + \cdots .
\] (2.119)

The Chern character is convenient for studying the direct sum and tensor product of vector bundles. For two vector bundles \( A \) and \( B \) based on the same manifold \( M \), the following hold [8]:

\[
\text{ch}(A \oplus B) = \text{ch}(A) + \text{ch}(B), \quad \text{and} \quad \text{ch}(A \otimes B) = \text{ch}(A)\text{ch}(B) .
\] (2.120)

3. Splitting Principle

The splitting principle is widely used to manipulate algebraic identities involving characteristic classes. It allows us to perform characteristic class operations as if every bundle is a direct sum of 1-dimensional line bundles. When a complex vector bundle \( E \) is split into a direct sum of \( k \) complex line bundles \( L_j \), each having curvature \( \Omega_j \), ie.

\[
E = \bigoplus_{j=1}^k L_j
\] (2.121)

then

\[
c(L_j) = 1 + x_j, \quad x_j = \frac{i}{2\pi} \Omega_j ,
\] (2.122)

and the total Chern form is

\[
c(E) = \prod_{j=1}^k \left( 1 + \frac{i}{2\pi} \Omega_j \right) = \prod_{j=1}^k (1 + x_j) .
\] (2.123)

In comparison with Eq. (2.115), it is equivalent to diagonalize the \( k \times k \) matrix 2-form \( \Omega \) into a direct sum of \( k \) curvature two-forms \( \Omega_j \) (\( j = 1, \cdots, k \)) of complex line bundles,

\[
(\Omega_j') = \bigoplus_{j=1}^k \Omega_j = \begin{pmatrix}
\Omega_1 & & \\
& \Omega_2 & \\
& & \ddots \ \\
& & & \Omega_k
\end{pmatrix}.
\]

Each Chern class takes an elegant form,

\[
c_1(E) = \sum_{j=1}^k x_j, \quad c_2(E) = \sum_{i<j} x_i x_j, \quad \cdots, \quad c_k(E) = x_1 x_2 \cdots x_k .
\] (2.124)
For the dual bundle $E^\ast$, there exists $E = \bigoplus_{j=1}^k L^\ast_j$. Since $c(L^\ast_j) = 1 - x_j$, so we obtain

$$C(E^\ast) = \prod_{j=1}^k (1 - x_j).$$

(2.125)

With the splitting principle, characteristic classes are conveniently defined by their generating functions and expressed as combinations of Chern classes. For example, the total Chern class is generated by $\prod (1 + x_j)$ and the Chern character generated by $\sum e^{x_j}$. Another class is the Todd class which is generated by

$$td(E) = \prod_{j=1}^k \frac{x_j}{1 - e^{-x_j}} = 1 + \frac{1}{2} c_1(E) + \frac{1}{12} (c_1^2 + c_2)(E) + \cdots .$$

(2.126)

The Todd class has the following property: for two vector bundles $A$ and $B$, there exists

$$td(A \oplus B) = td(A)td(B).$$

(2.127)

2.3.2 Pontrjagin Class

The Pontrjagin classes are defined for a $k$-dimensional real vector bundle $E$ with transition functions from $GL(k, \mathbb{R})$. As mentioned in Sect. 2.2, when a metric is introduced on the fibre, $GL(k, \mathbb{R})$ reduces to the orthogonal group $O(k)$. In fact, it can be proved that the characteristic classes corresponding to $GL(k, \mathbb{R})$ and $O(k)$ differ just by an exact form [8], so they lead to the same cohomology class. Therefore, we just take the structure group $G = O(k)$, consequently the curvature 2-form satisfies $\Omega = -\Omega^t$ since $\Omega$ is a real 2-form taking value in the Lie algebra of $O(k)$.

The total Pontrjagin class is defined by the invariant polynomial

$$p(E) = \det \left( I - \frac{1}{2\pi} \Omega \right) = 1 + p_1(\Omega) + p_2(\Omega) + \cdots = 1 - \frac{1}{8\pi^2} \text{Tr} \Omega^2 + \frac{1}{(2\pi)^4} \left[ \frac{1}{8} \left( \text{Tr} \Omega^2 \right)^2 - \frac{1}{4} \text{Tr} \Omega^4 \right] + \cdots ,$$

(2.128)

where we have used $\text{Tr} (\Omega^{2p+1}) = 0$ for an antisymmetric matrix $\Omega$. Clearly,

$$p_j(\Omega) \in H^{4j}(M),$$

(2.129)

and when $4j > n$ or $2j > k$, $p_j(\Omega) = 0$.

In the following we derive the generating function of $P(E)$. First, the Pontrjagin class can be expressed in terms of the Chern class by complexifying a real bundle $E$ as $E_c \equiv E \otimes \mathbb{C}$. Then comparing $P(E)$ with $C(E_c)$ where,

$$C(E_c) = \det \left( I + \frac{i}{2\pi} \Omega \right) = 1 + c_1(\Omega) + c_2(\Omega) + \cdots = 1 - p_1(\Omega) + p_2(\Omega) - \cdots ,$$

(2.130)

we have

$$p_j(E) = (-1)^j c_{2j}(E_c).$$

(2.131)
Conversely, a $k$-dimensional complex vector bundle $E$ can be considered as a $2k$-dimensional complex vector bundle $E_r$ by leaving out the complex structure of $E$. Complexify $E_r$ as $(E_r)_c = E_r \otimes \mathbb{C}$, $(E_r)_c$ becomes a complex bundle of $2k$ complex dimensions. Let $\overline{E}$ denote the complex conjugate bundle of the original complex bundle $E$. Then $\overline{E}$ is isomorphic to $E^* [8]$, the dual bundle of $E$. Hence

$$ (E_r)_c = E \oplus \overline{E} = E \oplus E^* = \left( \bigoplus_{j=1}^{k} L_j \right) \oplus \left( \bigoplus_{j=1}^{k} L_j^* \right). \tag{2.132} $$

From Eqs. (2.123), (2.117), (2.118), (2.130) and (2.132), we have

$$ C[(E_r)_c] = 1 - p_1(E_r) + p_2(E_r) - \ldots = C(E) \wedge C(\overline{E}) $$

$$ = C(E) \wedge C(E^*) = \prod_{j=1}^{k} \left( 1 - x_j^2 \right). \tag{2.133} $$

Hence we obtain the following generating function of $p_j(E_r)$,

$$ P(E) = P(E_r) = 1 + p_1(E_r) + p_2(E_r) + \cdots = \prod_{j=1}^{k} \left( 1 + x_j^2 \right), $$

$$ p_1(E) = \sum_i x_i^2, \quad p_2(E) = \sum_{i<j} x_i^2 x_j^2, \quad \ldots \quad p_k(E) = x_1^2 x_2^2 \cdots x_k^2. \tag{2.134} $$

Two other characteristic classes can be defined from the following generating functions, both of which are related the Pointrjagin class:

1. **Hirzebruch L-polynomial**:

   $$ L(E) = \prod_j \frac{x_j}{\tanh x_j}. \tag{2.135} $$

2. **$\hat{A}$ genus polynomial**:

   $$ \hat{A}(E) = \prod_j \frac{x_j/2}{\sinh(x_j/2)}. \tag{2.136} $$

### 2.3.3 Euler Class

The **Euler class** is defined for an oriented, real vector bundle $E$ of $k = 2r$ dimensions. The transition functions of the vector bundle belong to $SO(k)$, and the curvature 2-form $\Omega$ takes value in the Lie algebra of $SO(k)$. Hence $\Omega$ is a $k \times k$ antisymmetric matrix in the fibre space,

$$ \Omega_{ij} = -\Omega_{ji} = \frac{1}{2} \Omega_{\nu \rho ij} dx^\mu \wedge dx^\nu. \tag{2.137} $$

The Euler class $e(E)$ can be defined as a $k$-form that satisfies

$$ e(E) \wedge e(E) = p_{k/2}(\Omega). \tag{2.138} $$
In this way, we obtain an $SO(k)$-invariant polynomial in terms of $\Omega$. Clearly, because $p_{k/2}(M) \in H^{2k}(M,R)$, $k = 2r$, and $dp_{k/2}(\Omega) = 2[de(E)] \wedge e(E) = 0$ we have,

$$e(E) \in H^k(M,R).$$

The $SO(k)$ invariance of $e(E)$ can be observed in the following way [8]: use $\Omega_{ij}$ to formally construct a “two-form” in the fibre space,

$$\alpha = \frac{1}{2} \Omega_{ij} dx^i \wedge dx^j,$$

(2.140)

with $z^i, i = 1, \cdots, k = 2r$ being the coordinates of fibre space. Then define the $r = k/2$-fold wedge product to get

$$e(E) = 1 \times \cdots \times \frac{1}{r!} \left( \frac{\alpha}{2\pi} \right)^r = \frac{1}{r!} \left( \frac{\alpha}{4\pi} \right)^r \Omega_{i_1 j_1} \cdots \Omega_{i_r j_r} dz^{i_1} \wedge dz^{j_1} \wedge \cdots \wedge dz^{i_r} \wedge dz^{j_r}$$

(2.141)

$$= \frac{1}{r!} \left( \frac{\alpha}{4\pi} \right)^r \Omega_{i_1 j_1} \cdots \Omega_{i_r j_r} e^{i_1 j_1 + \cdots + j_r} dz^1 \wedge \cdots \wedge dz^k = e(E) dz^1 \wedge \cdots \wedge dz^k.$$

For a $k \times k$ transition matrix $g(x) \in SO(k)$, $\det g(x) = 1$ so (2.141) shows that $e(\Omega)$ is explicitly $SO(k)$ invariant.

Using the splitting principle, we put $\Omega$ into a standard form,

$$\Omega = -\Omega^t \rightarrow \begin{pmatrix} 0 & x_1 \\ -x_1 & 0 & \ddots \\ & & 0 & x_r \\ & & -x_r & 0 \end{pmatrix}.$$

Then from Eq. (2.140) and (2.141), we obtain

$$\alpha = x_1 dz^1 \wedge dz^2 + \cdots + x_r dz^{2r-1} \wedge dz^{2r},$$

and hence

$$e(E) = x_1 x_2 \cdots x_r = \prod_{j=1}^{k/2} x_j = (p_k)^{1/2}.$$

(2.142)

Note that the Euler class can only be calculated with the Riemannian connection since it is $SO(k)$ invariant rather than $GL(k,R)$ invariant. The Euler class $e(M)$ for a differential manifold $M$ represents the Euler class of the tangent bundle $T(M)$. Also $e(E)=0$ for odd-dimensional bundles. Like the Pontrajagin class, the Euler class for a direct sum of vector bundles $A$ and $B$ satisfies

$$e(A \oplus B) = e(A) \wedge e(B).$$

### 2.3.4 Stiefel-Whitney Class

The Stiefel-Whitney (SW) classes for a real bundle $E$ with a $k$-dimensional fibre over an $n$-dimensional base manifold $M$ is the $\mathbb{Z}_2$ cohomology classes of $M$:

$$w_i \in H^i(M; \mathbb{Z}_2), \quad i = 1, \cdots, n - 1,$$

(2.143)
and
\[ w_n \in H^n(M; \mathbb{Z}). \tag{2.144} \]

The total Stiefel-Whitney class is
\[ w(E) = 1 + w_1 + \cdots + w_{n-1} + w_n. \tag{2.145} \]

Note that the SW classes in general cannot be expressed as polynomials of the curvature. The Stiefel-Whitney classes are very useful for showing that how the topological property of the base manifold can determine the geometric structure. Some examples of the applications are [8]:

- \( w_1[T(M)] = 0 \) if and only if \( M \) is orientable.
- Spinors are well-defined on \( M \) and \( M \) is a spin-manifold if and only if \( w_1[T(M)] = w_2[T(M)] = 0 \).
- If \( w_2[T(M)] \neq 0 \), \( M \) does not admit a spin structure.

2.4 Classical Elliptic Complexes and Index Theorems

2.4.1 Elliptic Complexes and Index of Elliptic Differential Operator

1. Elliptic Differential Operator

Let us begin this section by letting \( M \) be a compact, smooth, \( n \)-dimensional manifold without boundary, with local coordinates \( \{x_j\} \). Let \( E \) and \( F \) be two vector bundles over \( M \) of dimension \( k \) and \( l \), and let \( \Gamma(E) \) and \( \Gamma(F) \) denote the sets of the sections of \( E \) and \( F \). Then, in terms of the coordinates at each point \( x \in M \) we have,
\[
\Gamma(E) = \left\{ \left( \chi^1(x), \ldots, \chi^k(x) \right) \right\}, \quad \text{and} \quad \Gamma(F) = \left\{ \left( \eta^1(x), \ldots, \eta^l(x) \right) \right\}. \tag{2.146}
\]

A differential operator \( D : \Gamma(E) \to \Gamma(F) \) can be expressed as the following form in terms of local coordinates,
\[
D \left( \chi^1(x), \ldots, \chi^k(x) \right) = \left( \eta^1(x), \ldots, \eta^l(x) \right),
\eta^j(x) = D^i_j \chi^j(x) = \sum_{j,p_1,\ldots,p_n} a^i_{j,p_1\ldots,p_n}(x) \frac{\partial^{p_1+\cdots+p_n}}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}} \chi^j(x),
\]
\[ p_1 + \cdots + p_n \leq p, \quad i = 1, \ldots, k, \quad j = 1, \ldots, l. \tag{2.147} \]

By performing the Fourier transform of the section \( \chi(x) \),
\[
\chi(x) = \int dk_1 \cdots dk_n e^{ik_1x_1+\cdots+k_nx_n} \chi(k),
\]
we can define from Eq. (2.147) the symbol \( \sigma(D) \) of \( D \),
\[
\eta = D\chi = \int d^n k e^{ik \cdot x} \sum_{p_1,\ldots,p_n} \sigma(D)(x,k) \chi(k),
\sigma(D)(x,k) = \sum_{p_1+\cdots+p_n \leq p} (a_{p_1\ldots,p_n}(x)) k_1^{p_1} \cdots k_n^{p_n}. \tag{2.148}
\]
In $\sigma(D)(x,k)$, the terms with the highest order in $k$ form the leading symbol of $D$, denoted by $\tilde{D}(x,k)$,

$$\tilde{D}(x,k) = \sum_{p_1+\cdots+p_n=p} a_{p_1} \cdots a_{p_n} x^{k_1} \cdots x^{k_n}.$$  

(2.149)

$D$ is called an elliptic operator if $\dim \Gamma(E) = \dim \Gamma(F)$ and $\tilde{D}(x,k)$ is invertible for $k \neq 0$, i.e.,

$$\det \tilde{D}(x,k) \neq 0 \quad \text{for} \quad k_1, \cdots, k_n \neq 0.$$

2. Analytic Index of Elliptic Differential Operator and Elliptic Complex

The kernel, image and cokernel of $D$ are defined as follows:

$$\text{Ker}D = \{ \chi \in \Gamma(E) : D\chi = 0 \}, \quad \text{(2.150)}$$

$$\text{Im}D = \{ \eta \in \Gamma(F) : \eta = D\chi, \chi \in \Gamma(E) \}, \quad \text{(2.151)}$$

$$\text{Coker}D = \Gamma(F) - \text{Im}D \equiv \Gamma(F)/\text{Im}D. \quad \text{(2.152)}$$

Then the analytic index of $D$ is given by

$$\text{Index}(D) = \dim \text{Ker}(D) - \dim \text{Coker}(D). \quad \text{(2.153)}$$

Further, because $D : \Gamma(E) \to \Gamma(F)$ is a linear invertible operator from the vector space $\Gamma(E)$ to the vector space $\Gamma(F)$ we have from elementary algebra [10]

$$\dim \Gamma(E) = \dim \text{Ker}D + \dim \text{Im}D, \quad \text{and}$$

$$\dim \Gamma(F) = \dim \text{Coker}D + \dim \text{Im}D. \quad \text{(2.154)}$$

Hence from (2.153)

$$\text{Index}(D) = \dim \Gamma(E) - \dim \Gamma(F). \quad \text{(2.155)}$$

When there exist metric structures in both fibre spaces of $E$ and $F$, an adjoint operator $D^\dagger : \Gamma(F) \to \Gamma(E)$ can be defined with the inner product given by the metrics,

$$(D\chi, \eta) = \left(\chi, D^\dagger\eta\right). \quad \text{(2.156)}$$

Then two non-negative self-adjoint elliptic operators can be constructed as,

$$\Delta_E = D^\dagger D : \Gamma(E) \to \Gamma(E), \quad \text{and}$$

$$\Delta_F = DD^\dagger : \Gamma(F) \to \Gamma(F). \quad \text{(2.157)}$$

There exists the following result:

**Theorem 2.4.1:** Let

$$\Gamma_\lambda(E) \equiv \{ \chi \in \Gamma(E) | \Delta_E \chi = \lambda \chi \}, \quad \text{and}$$

$$\Gamma_\lambda(F) \equiv \{ \eta \in \Gamma(F) | \Delta_F \eta = \lambda \eta \}. \quad \text{(2.158)}$$

Then for $\lambda \neq 0$, $\Gamma_\lambda(E)$ is isomorphic to $\Gamma_\lambda(F)$, and further, there exist

$$\Gamma_0(E) = \text{Ker}D, \quad \text{and} \quad \Gamma_0(F) = \text{Ker}D^\dagger. \quad \text{(2.159)}$$
**Proof:** Let $\chi_\lambda \in \Gamma_\lambda(E)$, $\lambda \neq 0$. Then by definition of $\Gamma_\lambda(E)$, $\Delta_E \chi_\lambda = D^\dagger D \chi_\lambda = \lambda \chi_\lambda$. Hence

$$\Delta_F(D \chi_\lambda) = DD^\dagger D \chi_\lambda = \lambda(D \chi_\lambda).$$

Thus $D \chi_\lambda \equiv \eta_\lambda \in \Gamma_\lambda(F)$ and moreover, $D \chi_\lambda \neq 0$. Otherwise, there would arise $\Delta_E \chi_\lambda = 0$ and this contradicts with $\lambda \neq 0$. Therefore, $D : \Gamma_\lambda(E) \rightarrow \Gamma_\lambda(F)$ is a one-to-one correspondence for $\lambda \neq 0$. Similarly, $D^\dagger : \Gamma_\lambda(F) \rightarrow \Gamma_\lambda(E)$ is also a bijective map. Hence $\Gamma_\lambda(E) \cong \Gamma_\lambda(F)$, and $\dim \Gamma_\lambda(E) = \dim \Gamma_\lambda(F)$.

Theorem 2.4.1 implies

$$\dim \text{Im} D = \dim \text{Im} D^\dagger. \quad (2.160)$$

Similar to $\dim \Gamma(E)$ in (2.154), there exists

$$\dim \Gamma(F) = \dim \text{Ker} D^\dagger + \dim \text{Im} D^\dagger = \dim \text{Ker} D^\dagger + \dim \text{Im} D = \dim \text{Coker} D + \dim \text{Im} D. \quad (2.161)$$

Hence $\dim \text{Coker} D = \dim D^\dagger$. Therefore, from Eq. (2.153), vector bundles which have metric structures in their fibre spaces have the following analytic index of the elliptic operator,

$$\text{Index}(D) = \dim \text{Ker}(D) - \dim \text{Ker} D^\dagger. \quad (2.162)$$

### 3. Elliptic Complex (The Sequence Formulation)

Let $\{E_p\}$ be a finite sequence of vector bundles over $M$. Also, let $\{D_p\}$ be the sequence $D_p : \Gamma(E_p) \rightarrow \Gamma(E_{p+1})$ of differential operators as defined above. The pair $(E, D) = (\{E_p\}, \{D_p\})$ is a **complex** if $D_{p+1}D_p = 0$. Let $D_p^\dagger : \Gamma(E_{p+1}) \rightarrow \Gamma(E_p)$ be the adjoint operator of $D$. Define the corresponding Laplacian as

$$\Delta_p = D_p^\dagger D_p + D_p^{-1}D_{p-1}^\dagger. \quad (2.163)$$

$(E, D)$ is an **elliptic complex** if $\Delta_p$ is an elliptic operator on $\Gamma(E_p)$. Then the cohomology groups for $(E, D)$ are given by

$$H^p(E, D) = \frac{\text{Ker} D_p}{\text{Im} D_{p-1}} \cong \text{Ker} \Delta_p. \quad (2.164)$$

The analytic index of $(E, D)$ is

$$\text{Index}(E, D) = \sum_p (-1)^p \dim H^p(E, D) = \sum_p (-1)^p \dim \text{Ker} \Delta_p. \quad (2.165)$$

### 4. Two-grade formulation of Elliptic Complex

An elliptic complex has another equivalent two-grade formulation. Define the even and odd bundles as

$$E^o = \bigoplus_p E_{2p}, \quad \text{and} \quad E^o = \bigoplus_p E_{2p+1}. \quad (2.166)$$
Then combine the operators as
\[
\mathcal{O} = \bigoplus_p \left( D_{2p} + D_{2p-1}^\dagger \right), \quad \text{and} \quad \mathcal{O}^* = \bigoplus_p \left( D_{2p}^\dagger + D_{2p+1} \right).
\]
(2.167)

Clearly, there exists mappings \( O : \Gamma(E^e) \rightarrow \Gamma(E^o) \) and \( O : \Gamma(E^o) \rightarrow \Gamma(E^e) \). The associated Laplacian operators are
\[
\Delta_e = O^* \mathcal{O} = \bigoplus_q \Delta_{2q}, \quad \text{and} \quad \Delta_o = \mathcal{O} O^* = \bigoplus_q \Delta_{2q+1}.
\]
(2.168)

Consequently, the complex is \((E, \mathcal{O}) \equiv (\{E^e, E^o\}, \{O, O^*\})\), and the index of the complex is the same as that described in the sequence form
\[
\text{Index}(E, O) = \dim \ker \Delta_e - \dim \ker \Delta_o = \sum_p (-1)^p \dim \ker \Delta_p = \text{Index}(E, D).
\]
(2.169)

5. Atiyah-Singer Index Theorem for Elliptic Complex

Now we are able to define the general form of the Atiyah-Singer index theorem for elliptic complexes. Let \( \text{ch}(E_p) \) be the Chern character of \( E_p \), \( \text{td}[T(M)] \) be the Todd class of the tangent bundle, and \( e(M) \) be the Euler class of \( T(M) \). Then the Atiyah-Singer index theorem \([8]\) states
\[
\text{Index}(E, D) = (-1)^{n(n+1)/2} \int_M \text{ch} \left( \bigoplus_p (-1)^p E_p \right) \frac{\text{td}[T(M) \otimes C]}{e(M)}.
\]
(2.170)

This general form is applied to the four classical elliptic complexes in the subsequent subsections.

2.4.2 de Rham Complex and Gauss-Bonnet Theorem

The de Rham complex follows from decomposing the exterior algebra \( \Lambda^*(M) \) into the following two-grade (even and odd) forms:
\[
\Lambda^{\text{even}} = \Lambda^0 \oplus \Lambda^2 \oplus \ldots, \quad \Lambda^{\text{odd}} = \Lambda^1 \oplus \Lambda^3 \oplus \ldots.
\]
(2.171)

Denote the de Rahm complex as \((\Lambda^{\text{even/odd}}, d + \delta)\) with operator \( d + \delta \) such that
\[
d + \delta : \Gamma(\Lambda^{\text{even}}) \rightarrow \Gamma(\Lambda^{\text{odd}}).
\]
(2.172)

This gives the de Rham cohomology group \( H^p(M, R) = \ker d_p / \text{im} d_{p-1} \) with the analytic index of \((\Lambda^{\text{even/odd}}, d + \delta)\) to be the Euler characteristic
\[
\text{Index}(\Lambda^{\text{even/odd}}, d + \delta) = \sum_{p=0}^n (-1)^p \dim H^p(M, R) = \chi(M).
\]
(2.173)

Next apply the general form (2.170) to the de Rham complex. In the de Rham complex,
\[
E_p = \Lambda^p = \bigwedge_p [T^*(M)]^p = T^*(M) \wedge \cdots \wedge T^*(M).
\]
(2.174)
Elliptic Complexes

By the splitting principle,

$$T^*(M) = \bigoplus_{j=1}^n L_j^*.$$  \hfill (2.175)

Hence from (2.174),

$$\Lambda^p = \bigwedge [T^*(M)]^p = \bigwedge \left( \bigoplus_{j=1}^n L_j^* \right)^p = \bigoplus_{1 \leq j_1 < j_2 < \cdots < j_p \leq n} (L_{j_1} \otimes \cdots \otimes L_{j_p}).$$  \hfill (2.176)

Using the property of the Chern character for the direct sum and tensor product of vector bundles given in (2.120) and the result for a dual line bundle, \(\text{ch}(L_j) = e^{-x_j}\), we obtain

$$\text{ch}(\Lambda^p) = \text{ch} \left( \bigoplus_{1 \leq j_1 < j_2 < \cdots < j_p \leq n} (L_{j_1} \otimes \cdots \otimes L_{j_p}) \right) = \sum_{1 \leq j_1 < j_2 < \cdots < j_p \leq n} e^{-(x_{j_1} + x_{j_2} + \cdots + x_{j_p})}.$$  \hfill (2.177)

The result (2.177) leads to

$$\text{ch} \left( \bigoplus_{p=0}^{n} (-1)^p \Lambda^p \right) = \sum_{p=0}^{n} (-1)^p \sum_{1 \leq j_1 < j_2 < \cdots < j_p \leq n} e^{-(x_{j_1} + x_{j_2} + \cdots + x_{j_p})} = \prod_{j=1}^{n} (1 - e^{-x_j}).$$  \hfill (2.178)

Further, from Eq. (2.126), (2.127) and (2.142),

$$\text{td}[T(M) \otimes C] = \text{td} \left[ T(M) \oplus \overline{T(M)} \right] = \text{td} [T(M) \oplus T^*(M)] = \text{td} [T(M)] \text{td} [T^*(M)] = \prod_{j=1}^{n/2} \left[ \frac{x_j}{1 - e^{-x_j}} \right],$$  \hfill (2.179)

and

$$e(M) = \prod_{j=1}^{n/2} x_j.$$  \hfill (2.179)

Substituting (2.178) and (2.179) into (2.170), we obtain the Gauss-Bonnet theorem,

$$\text{Index}(\Lambda^\text{even/odd}, d + \delta) = \chi(M) = (-1)^{n(n+1)/2} \int_M \text{ch} \left( \bigoplus_{p=0}^{n} (-1)^p \Lambda^p \right) \frac{\text{td}[T(M) \otimes C]}{e(M)}$$

$$= (-1)^{n(n+1)/2} (-1)^{n/2} \int_M \prod_{j=1}^{n/2} x_j = \int_M e(M),$$  \hfill (2.180)

where \(n\) is an even number.
2.4.3 Signature Complex and Hirzebruch Signature Theorem

The signature complex is based on a different splitting of the exterior algebra $\Lambda^*(M)$ on an $n = 4k$-dimensional Riemannian manifold. Let $\omega$ be an operator action on $p$-forms defined by

$$\omega = i^{p(p-1)+n/2} \ast.$$  \hspace{1cm} (2.181)

It can easily be proven that

$$\omega d = -\delta \omega, \quad \omega \delta = -d \omega, \quad \text{and} \quad \omega^2 = 1.$$  \hspace{1cm} (2.182)

Hence there arises

$$\omega (d + \delta) = -(d + \delta) \omega.$$  \hspace{1cm} (2.183)

When $n = 4k$, $\omega = \ast$ on $\Lambda^{2k}$. Letting $\Lambda^\pm$ be the $\pm 1$ eigenspaces of $\omega$, we have

$$\Lambda^{p+} = \{ \alpha_p \in \Lambda^p : \omega \alpha_p = \alpha_p \}, \quad \Lambda^{p-} = \{ \alpha_p \in \Lambda^p : \omega \alpha_p = -\alpha_p \},$$

$$\Lambda^+ = \oplus \Lambda^{p+}, \quad \Lambda^- = \oplus \Lambda^{p-}, \quad \Lambda = \Lambda^+ \oplus \Lambda^-.$$  \hspace{1cm} (2.184)

From Eq. (2.183), there exists $d + \delta : \Lambda^+ \rightarrow \Lambda^-$. \hspace{1cm} (2.185)

$(\Lambda^\pm, d + \delta)$ is called the signature complex. The analytic index of the signature complex is

$$\text{Index}(\Lambda^\pm, d + \delta) = \dim H^{2k}_+ (M, \mathbb{R}) - \dim H^{2k}_- (M, \mathbb{R})$$

$$= b^+_2 - b^-_2 = \tau(M),$$  \hspace{1cm} (2.186)

which is contributed to only by the harmonic $p$-forms with $p = n/2 = 2k$.

We apply the general form (2.170) of the Atiyah-Singer index theorem to the signature complex, for which $\{ E_p \} = \{ \Lambda^+, \Lambda^- \}$. It can be found in a similar way as deriving (2.178) that

$$\text{ch} \left( \bigoplus_p E_p \right) = \text{ch} (\Lambda^+) - \text{ch} (\Lambda^-) = \prod_{j=1}^{n/2} (e^{-x_j} - e^{x_j}).$$  \hspace{1cm} (2.187)

Substituting (2.178) and (2.187) into (2.170), we obtain we obtain the Hirzebruch signature theorem

$$\tau(M) = (-1)^{n(n+1)/2} \int_M \left( \text{ch} [\Lambda^+] - \text{ch} [\Lambda^-] \right) \frac{\text{td}[T(M) \otimes \mathbb{C}]}{e(M)}$$

$$= (-1)^{n(n+1)/2} \int_M \prod_{j=1}^{n/2} \left[ \frac{e^{-x_j} - e^{x_j}}{x_j} \frac{x_j}{1 - e^{-x_j}} \frac{-x_j}{1 - e^{x_j}} \right]$$

$$= \int_M \prod_{j=1}^{n/2} \frac{x_j (e^{x_j} - e^{-x_j})}{(e^{x_j/2} - e^{-x_j/2})^2} = \prod_{j=1}^{n/2} \frac{x_j \cosh(x_j/2)}{\sinh(x_j/2)}$$

$$= 2^{n/2} \int_M \prod_{j=1}^{n/2} \frac{x_j/2}{\tanh(x_j/2)} = \int_M L(M),$$  \hspace{1cm} (2.188)

where $L(M)$ is the Hirzebruch L-polynomial given in Eq. (2.135) (up to a constant factor). If $n \neq 4k$, then $\tau(M) = 0$. 

2.4.4 Dolbeault Complex and Riemann-Roch Theorem

Let $M$ be a complex manifold with real dimension $n = 2k$. From Subsection 2.1.5, the exterior algebra splits into the sets $\Lambda^{p,q}$ and there exists the antiholomorphic exterior differential operator,

$$\overline{\partial} = d\overline{z} \frac{\partial}{\partial z}, \quad \Gamma(\Lambda^{p,q}) \longrightarrow \Gamma(\Lambda^{p,q+1}).$$  \hspace{1cm} (2.189)

The Dolbeault complex is $\left(\{\Lambda^{0,q}\}, \overline{\partial}\right)$, and the analytic index is

$$\text{Index}(\overline{\partial}) = \sum_{q=0}^{n/2} (-1)^q \dim H^{0,q}(M).$$  \hspace{1cm} (2.190)

For the Dolbeault complex, $E_p$ is played by $\Lambda^{0,q}$. In a similar way as deriving (2.178), we can find

$$\text{ch} \left( \bigoplus (-1)^q \Lambda^{0,q} \right) = \prod_{j=1}^{n/2} \left( 1 - e^{-x_j} \right).$$  \hspace{1cm} (2.191)

Substituting (2.178) and (2.191) into (2.170), we obtain the Riemann-Roch theorem:

$$\text{Index}(\overline{\partial}) = (-1)^{n(n+1)/2} \int_M \text{ch} \left( \bigoplus (-1)^q \Lambda^{0,q} \right) \frac{\text{td}[T(M) \otimes \mathbb{C}]}{e(M)}$$

$$= (-1)^{n(n+1)/2} \int_M \prod_{j=1}^{n/2} \left[ \frac{1 - e^{-x_j}}{x_j} \right] \int_M \text{td}[T_c(M)],$$  \hspace{1cm} (2.192)

where $T_c(M)$ is the complex tangent space defined in (2.72).

2.4.5 Spin Complex and Index Theorem for $A$-roof Genus

The spin complex is defined as follows: First, as mentioned in Sect. 2.2 and shown in Appendix C, the sections of the spin bundle are spinor functions $\psi(x)$, and the set of sections can decompose into two subspaces composed of chiral spinor functions,

$$S = S^+ \oplus S^-,$$

$$S^+ = \{ \psi_+(x) \in S : \gamma_{n+1} \psi_+(x) = \psi_+(x) \},$$

$$S^- = \{ \psi_-(x) \in S : \gamma_{n+1} \psi_-(x) = -\psi_-(x) \}.$$  \hspace{1cm} (2.193)

The operator acting on the sections of spin bundles is the Dirac operator defined in (2.110),

$$D = \gamma^a E_a^\mu D_\mu.$$  \hspace{1cm} (2.194)

Because $\gamma_{n+1} \gamma_a = -\gamma_a \gamma_{n+1}$, there arises

$$\gamma_{n+1} D = -D \gamma_{n+1},$$  \hspace{1cm} (2.195)

$$D : \Gamma(S^+) \longrightarrow \Gamma(S^-), \text{ and}$$

$$D^\dagger : \Gamma(S^-) \longrightarrow \Gamma(S^+).$$  \hspace{1cm} (2.196)
Atiyah-Singer Index Theorem

\[(D, S^\pm)\) is the spin complex and the analytic index of the spin complex is

\[
\text{Index}(S^\pm, D) = \dim \text{Ker} D - \dim \text{Ker} D^\dagger = n_+ - n_-, \tag{2.197}
\]

where \(n_\pm\) are the numbers of spinor functions with chirality \(\pm 1\). These are eigenfunctions of the Dirac operator corresponding to the zero eigenvalue, respectively.

We apply the general form of the Atiyah-Singer index theorem with \(\dim M = n = 4k\) on the spin complex, and \(\{E_p\} = \{S^\pm\}\). Using the results

\[
\text{ch}(S^+) - \text{ch}(S^-) = \prod_{j=1}^{n/2} \left( e^{x_j/2} - e^{-x_j/2} \right), \tag{2.198}
\]

with the same \(\text{td}[T(M)]\) and \(e(M)\) as shown in Eqs. (2.187) and (2.191), we obtain the index theorem for \(\hat{A}\) genus,

\[
\text{Index}(S^\pm, D) = (-1)^{n(n+1)/2} \int_M \left[ \text{ch}(S^+) - \text{ch}(S^-) \right] \frac{\text{td}[T(M) \otimes \mathbb{C}]}{e(M)}
\]

\[
= \int_M \prod_{j=1}^{n/2} \left( \frac{e^{x_j/2} - e^{-x_j/2}}{x_j} \frac{x_j}{1 - e^{-x_j}} - x_j \frac{1}{1 - e^{x_j}} \right)
\]

\[
= \int_M \prod_{j=1}^{n/2} \left( \frac{x_j/2}{\sinh(x_j/2)} \right) = \int_M \hat{A}(M), \tag{2.199}
\]

where \(\hat{A}(M)\) is the \(A\)-roof genus of tangent bundle on \(M\) given by

\[
\hat{A}(M) = \prod_{i=1}^{n/2} \frac{x_i/2}{\sinh(x_i/2)} = 1 - \frac{1}{24} p_1[T(M)] + \frac{1}{5760} (7p_1^2 - 4p_2)[T(M)] + \cdots .
\]
Chapter 3

Some Aspects of Supersymmetric Quantum Mechanics

Classical Mechanics describes the motion of a physical body. The most fundamental physical principles in classical mechanics are Newton’s three laws. With the invention of calculus of variations in mathematics, Newtonian mechanics has been developed into analytical mechanics: namely Lagrangian and Hamiltonian formulations. These two formulations have not only revealed the elegant algebraic and geometric structures of classical mechanical systems, but have also provided two equivalent routes to construct quantum theory from classical mechanics.

3.1 Lagrangian Formulation and Hamiltonian Formulation of Classical Mechanics

3.1.1 Lagrangian Mechanics

The fundamental principle underlying Lagrangian mechanics is the least action principle, while the mathematical foundation is the calculus of variations. Lagrangian mechanics is defined using a generalized position coordinate system \( q \equiv \{ q_a \} = \{ q_1, q_2, \cdots q_N \} \) of a mechanical system. The Lagrangian function \( L[q(t), \dot{q}(t)] \) is a function of the generalized position coordinates \( q \) and their derivatives \( \dot{q} = dq/dt \), the generalized velocities. The classical action is defined as

\[
S[q] = \int_{t_1}^{t_2} dt L[q(t), \dot{q}(t)].
\]

Geometrically, as \( t \) changes, \( q(t) \) describes a trajectory (or a path) in a configuration space \( \{ q_a \} \). The least action principle states that among all the possible trajectories joining \( q_1 \) at time \( t_1 \) to \( q_2 \) at time \( t_2 \), the actual physical one should be the trajectory that make \( S[q(t)] \) minimal, i.e.,

\[
\frac{\delta S}{\delta q_a(t)} = 0.
\]
Supersymmetric Quantum Mechanics

From

\[ \delta S = \int_{t_1}^{t_2} dt \left[ \frac{\partial L}{\partial q} \delta q_a + \frac{\partial L}{\partial \dot{q}_a} \delta \dot{q}_a \right] \]

\[ = \int_{t_1}^{t_2} dt \left\{ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_a} \right) + \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_a} \right) \right] \delta q_a \right\} \]

\[ = \int_{t_1}^{t_2} dt \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_a} \right) \right] \delta q_a, \quad (3.3) \]

we obtain the Euler-Lagrange equations describing the motion of the mechanical system,

\[ \frac{\delta S[q]}{\delta q(t)} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q} = 0. \quad (3.4) \]

In Eq. (3.3) we have used

\[ \delta \dot{q}_a(t) = \frac{d}{dt} \delta q_a(t), \quad \text{and} \quad \delta q(t_1) = \delta q(t_2) = 0. \quad (3.5) \]

For example, consider a particle of mass \( m \) moving in \( \mathbb{R}^3 \) governed by a potential field \( V(x, y, z) \). The generalized position coordinates are \( q = (x, y, z) \). The corresponding Lagrangian function is

\[ L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z). \]

The Euler-Lagrangian equation (3.4) gives

\[ m \ddot{x} + \frac{\partial V}{\partial x} = 0, \quad m \ddot{y} + \frac{\partial V}{\partial y} = 0, \quad \text{and} \quad m \ddot{z} + \frac{\partial V}{\partial z} = 0. \]

Written in vector notation, this represents \( m \ddot{q} = -\nabla V = \mathbf{F} \), which is exactly the equation of motion determined by Newton’s second law.

### 3.1.2 Hamiltonian Mechanics and Poisson Bracket Operation

In Hamiltonian mechanics, we use a Hamiltonian to describe a classical mechanical system. To construct the Hamiltonian, we first find canonical conjugate momenta \( p_a \) conjugating the generalized position coordinates \( q_a \), defined by

\[ p_a = \frac{\partial L[q, \dot{q}]}{\partial \dot{q}_a}. \quad (3.6) \]

In physics, the \( 2N \) variables \((q, p) = (q_a, p_a)\) form a set that is called a phase space.

A Hamiltonian \( H[q(t), p(t)] \) is a function of \( q_a \) and \( p_a \) obtained from the Lagrangian function through a Legendre transformation

\[ H(q_a, p_a) = p_a \dot{q}_a(p, q) - L[q, \dot{q}_a(p, q)]. \quad (3.7) \]

Then the classical action (3.1) takes the form

\[ S[q, p] = \int_{t_1}^{t_2} dt \; L[q, \dot{q}] = \int_{t_1}^{t_2} dt \left[ p_a dq_a - H(p, q) \right]. \quad (3.8) \]
The least action principle gives
\[ \delta S[q, p] = \int_{t_1}^{t_2} dt \left[ \delta p_a \left( \dot{q}_a - \frac{\partial H}{\partial p_a} \right) + \left( p_a \frac{d\delta q_a}{dt} - \frac{\partial H}{\partial q_a} \delta q_a \right) \right] dt \]
and hence we obtain a set of Hamilton’s equations,
\[ \dot{q}_a = \frac{\partial H}{\partial p_a}, \quad \text{and} \quad \dot{p}_a = -\frac{\partial H}{\partial q_a}. \quad (3.10) \]

For the above example considered in the Lagrangian formulation, the conjugate momenta are
\[ p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad \text{and} \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}. \]
The corresponding Hamiltonian is
\[ H = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - L = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + V(x, y, z). \]
Then from Hamilton’s equations we have
\[ \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \quad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \quad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}, \]
and
\[ \dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{\partial V}{\partial y}, \quad \dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z}. \]
It is clear that these equations represent the classical equations \( ma = F = -\nabla V. \)

In the following we introduce an important algebraic operation in phase space, the Poisson bracket. Let us consider a function \( f(q, p) \) in phase space. Then
\[ \frac{df}{dt} = \frac{\partial f}{\partial q_a} \dot{q}_a + \frac{\partial f}{\partial p_a} \dot{p}_a = \frac{\partial H}{\partial p_a} \frac{\partial f}{\partial q_a} - \frac{\partial H}{\partial q_a} \frac{\partial f}{\partial p_a} = [H, f]_{PB} \quad (3.11) \]
where the operation of the Poisson bracket on two functions \( f(q_a, p_a) \) and \( g(q_a, p_a) \) on phase space is defined by
\[ [f, g]_{PB} = \frac{\partial f}{\partial q_a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q_a}. \quad (3.12) \]
It is easy to see that there exist
\[ [q_a, p_b]_{PB} = \delta_{ab}, \quad \text{and} \quad [q_a, q_b]_{PB} = [p_a, p_b]_{PB} = 0. \quad (3.13) \]

### 3.2 Construction of Quantum Mechanics from Classical Mechanics

Quantum mechanics can be constructed from classical mechanics by a procedure called quantization. There are two equivalent quantization approaches which are called canonical quantization and path integral quantization. The former is based on the Hamiltonian formulation of classical mechanics, and the latter on the Lagrangian formulation.
3.2.1 Canonical Quantization

Under canonical quantization, the motion state of a mechanical system is represented by a vector $|\psi\rangle$ in a Hilbert space. The coordinate $q$, momentum $p$, and hence any function defined in phase space are changed into the operators,

\[
q \to \hat{q}, \quad p \to \hat{p}, \quad f(q, p) \to \hat{f}(\hat{q}, \hat{p}),
\]

in the Hilbert space. The Poisson bracket (3.12) is replaced by the Lie bracket between two operators,

\[
[f, g]_{PB} \to [\hat{f}, \hat{g}] \equiv \hat{f}\hat{g} - \hat{g}\hat{f}.
\]

Hence, the fundamental Poisson bracket operations listed in (3.13) become

\[
[\hat{q}_a, \hat{p}_b] = i\delta_{ab}, \quad \text{and} \quad [\hat{q}_a, \hat{q}_b] = [\hat{p}_a, \hat{p}_b] = 0.
\]

The change of a physical state is governed by the Schrödinger equation

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}(\hat{q}, \hat{p}) |\Psi(t)\rangle.
\]

In physics, we usually consider the conservative mechanical system. In this case,

\[
|\Psi(t)\rangle = e^{-iEt}|\psi\rangle,
\]

and $|\psi\rangle$ is the eigenvector of $\hat{H}$ with eigenvalue $E$,

\[
\hat{H}|\psi\rangle = E|\psi\rangle.
\]

The eigenvalue equation (3.19) is also called the time-independent Schrödinger equation.

Eq. (3.16) represents formal operations between abstract operators. To do explicit calculations, we must choose concrete representations for abstract operators and state vectors. Only two representations need to be introduced for later use in this paper: coordinate representation and particle number representation.

1. Coordinate representation

Coordinate representation is defined by choosing the eigenvectors $\{|q\rangle\}$ of the coordinate operator $\hat{q}$ as the basis of the Hilbert space. Then every vector in the Hilbert space can be expanded in terms of $\{|q\rangle\}$. As a basis of the Hilbert space, $\{|q\rangle\}$ must be complete, ie.

\[
\int dq |q\rangle \langle q| = 1,
\]

where 1 denotes the unit matrix of infinite rank. Then every vector $|\Psi(t)\rangle$ admits the expansion

\[
|\Psi(t)\rangle = \int dq |q\rangle \langle q|\Psi(t)\rangle = \int dq |q\rangle \Psi(q, t),
\]

where the coordinate component

\[
\Psi(q, t) \equiv \langle q|\Psi(t)\rangle
\]
Quantization is usually called a wave function. In physics, $|\Psi(q,t)|^2 = \overline{\Psi}(q,t)\Psi(q,t)$ is interpreted as the probability density function that a particle appears in the position $q$ at time $t$. As a consequence, in coordinate representation,

$$\hat{q} = q.$$  \hfill (3.23)

Then the operator relations (3.16) determine that $\hat{p}$ should be a partial derivative operator,

$$\hat{p} = -i \frac{\partial}{\partial q}. \hfill (3.24)$$

By definition, the Hilbert space $\mathcal{H}$ is composed of square-integrable complex-valued functions $\psi(q)$,

$$\mathcal{H} = \left\{ (\psi(q)) \mid 0 \leq \int_{-\infty}^{\infty} dq \, |\psi(q)|^2 < \infty \right\}. \hfill (3.25)$$

When $|\psi(q)|$ is normalized, $\int dq |\psi(q)|^2 = 1$. For example, the expansion of $|q\rangle$ is

$$|q\rangle = \int dq' |q'\rangle \langle q'|q\rangle = \int dq' \psi(q',q)|q\rangle,$$  \hfill (3.26)

and the wave function corresponding to $|q\rangle$ is the Dirac delta function \[11\]

$$\psi(q') = \langle q'|q\rangle = \delta(q - q'), \hfill (3.27)$$

As for the momentum operator $\hat{p}$, if $|p\rangle$ is an eigenvector of the momentum operator $\hat{p}$ with eigenvalue $p$, then it admits the expansion in the basis $\{|q\rangle\} [11],

$$|p\rangle = \int dq |q\rangle \langle q|p\rangle = \int dq \psi(q,p),$$

$$\psi(q,p) \equiv \langle q|p\rangle = \frac{1}{(2\pi)^{1/2}} \exp(ipq), \quad \text{and}$$

$$\hat{p}|p\rangle = -i \frac{\partial}{\partial q}|p\rangle = -i \int dq \frac{\partial}{\partial q} \psi(q,p)|q\rangle = p|p\rangle. \hfill (3.28)$$

The Schrödinger equation becomes a partial differential equation,

$$i\frac{\partial}{\partial t} \Psi(q,t) = H(q,-i\partial/\partial q)\Psi(q,t). \hfill (3.29)$$

The solution to the above equation takes the form,

$$\Psi(q,t) = e^{-iEt} \psi(q), \quad \text{and} \quad H\psi(q) = E\psi(q). \hfill (3.30)$$

All the solutions of (3.30) form the Hilbert space.

In fact, from Eqs. (3.20), (3.27) and (3.28), one can derive

$$\int dp \, |p\rangle\langle p| = 1,$$  \hfill (3.31)

so the set of eigenvectors $\{|p\rangle\}$ is also complete. We can also choose $\{|p\rangle\}$ as the basis of the Hilbert space, which in physics is usually called the momentum representation. Further, Eqs. (3.27), (3.28) and (3.31) yield the identity,

$$\delta(q - q') = \langle q'|q\rangle = \int dp |q'\rangle\langle p|q\rangle = \frac{1}{2\pi} \int dp \exp[ip(q' - q)]. \hfill (3.32)$$
2. Particle number representation

The particle number representation is defined as follows: First, define operators $a$ and the adjoint $a^\dagger$ as

$$a = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \quad \text{and}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}).$$

(3.33) \hspace{1cm} (3.34)

It is easy to see

$$[a, a^\dagger] = 1.$$

(3.35)

The Hamiltonian operator can be expressed as

$$\hat{H}(\hat{q}, \hat{p}) = \hat{H}(a, a^\dagger).$$

(3.36)

Second, find a normalized eigenvector of $\hat{H}$ such that $\hat{H}$ has a minimal eigenvalue, which is usually denoted as $|0\rangle$. Then the Hilbert space $\mathcal{H}$ can be constructed as

$$\mathcal{H} = \text{span}\left\{ |n\rangle |n\rangle \equiv \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \right\}.$$

(3.37)

The following uses the theory of the one-dimensional harmonic oscillator to illustrate the representation theory and construction of the corresponding Hilbert space [11]. The Hamiltonian is

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2).$$

(3.38)

1. Coordinate representation

In coordinate representation, from Eqs. (3.23) and (3.24), the Hamiltonian (3.38) is a second-order differential operator,

$$\hat{H} = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{1}{2} q^2,$$

(3.39)

and the time-independent Schrödinger equation (3.19) is a second-order ordinary linear differential equation,

$$\left(-\frac{1}{2} \frac{d^2}{dq^2} + \frac{1}{2} q^2\right) \psi(q) = E \psi(q).$$

(3.40)

Eq. (3.40) can be converted into the Hermite equation corresponding to different discrete eigenvalues

$$E_n = \left(n + \frac{1}{2}\right), n = 0, 1, 2, \cdots,$$

(3.41)

with solutions [11]

$$\psi_n(q) = \frac{1}{(2^n n! \sqrt{\pi})^{1/2}} \exp\left(-\frac{1}{2} q^2\right) H_n(q),$$

(3.42)

where $H_n(q)$ is the Hermite polynomial [20],

$$H_n(q) = (-1)^n e^{q^2} \left( \frac{d}{dq} e^{-q^2} \right).$$

(3.43)
2. Particle number representation

In particle representation, the Hamiltonian operator becomes

\[ \hat{H} = \frac{1}{2} (a a^\dagger + a^\dagger a) = N + \frac{1}{2} \]  

(3.44)

where

\[ N = a^\dagger a. \]  

(3.45)

Now the eigenvalue problem of \( H \) reduces to constructing the eigenvector of \( N \). Let \( |\nu\rangle \) be an eigenvector of \( N \) with corresponding eigenvalue \( \nu \),

\[ N|\nu\rangle = \nu|\nu\rangle. \]  

(3.46)

Notice that from (3.35) and (3.45),

\[ [N, a] = -a, \quad \text{and} \quad [N, a^\dagger] = a^\dagger, \]  

(3.47)

so there exist

\[ N (a|\nu\rangle) = a(N - 1)|\nu\rangle = (\nu - 1)a|\nu\rangle, \]
\[ N (a^\dagger|\nu\rangle) = a^\dagger(N + 1)|\nu\rangle = (\nu + 1)a^\dagger|\nu\rangle. \]  

(3.48)

This means that \( a|\nu\rangle \) is an eigenvector of \( N \) with eigenvalue \( \nu - 1 \) and \( a^\dagger|\nu\rangle \) is an eigenvector of \( N \) with eigenvalue \( \nu + 1 \). Using mathematical induction proves that for \( n \in Z^+ \),

\[ N (a^n|\nu\rangle) = (\nu - n)a^n|\nu\rangle, \]
\[ N (a^\dagger n|\nu\rangle) = (\nu + n)a^\dagger n|\nu\rangle. \]  

(3.49)

Hence we have obtained the set of eigenvectors of \( N \)

\[ a|\nu\rangle, a^2|\nu\rangle, \ldots, a^n|\nu\rangle, \ldots, \]

corresponding to the eigenvalues \( \nu - 1, \nu - 2, \ldots, \nu - p, \ldots \). Similarly, the set of eigenvectors

\[ a^\dagger|\nu\rangle, a^{\dagger 2}|\nu\rangle, \ldots, a^{\dagger p}|\nu\rangle, \ldots, \]

corresponds to eigenvalues \( \nu + 1, \nu + 2, \ldots, \nu + p, \ldots \). Because \( N \) is a self-adjoint (or Hermitian) operator, \( N = N^\dagger \), all its eigenvalues must be non-negative. This means the minimal eigenvalue of \( N \) should be zero, and the decreasing sequence of eigenvectors should stop at a certain step. Therefore, \( \nu = n \) and

\[ a|0\rangle = 0. \]  

(3.50)

This means the collection of all eigenvalues of \( N \) is equal to the set of non-negative integers,

\[ N|n\rangle = n|\rangle, \quad n \in Z, \quad n \geq 0. \]  

(3.51)

Requiring that all the eigenvectors are normal,

\[ \langle n|n\rangle = 1, \quad n = 0, 1, 2, \ldots, \]  

(3.52)
one can prove
\[ a^\dagger |n\rangle = (n + 1)^{1/2} |n + 1\rangle, \quad a|n\rangle = n^{1/2} |n - 1\rangle, \]
\[ |n\rangle = \frac{1}{(n!)^{1/2}} a^\dagger n |0\rangle, \quad \langle m|n\rangle = \delta_{mn}. \] (3.53)

Hence we have obtained the Hilbert space in the particle number representation,
\[ \mathcal{H} = \text{span} \left\{ |n\rangle \mid |n\rangle = \frac{1}{(n!)^{1/2}} a^\dagger n |0\rangle, \quad a|0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad n \in \mathbb{Z}, n \geq 0 \right\}. \] (3.54)

In particle number representation, all the operators have infinite dimensional matrix representations realized on $|n\rangle$:
\[ \hat{H} = (H_{mn}), \quad H_{mn} = (n + \frac{1}{2}) \delta_{m,n}, \]
\[ a = (a_{mn}), \quad a^\dagger = (a^\dagger_{mn}), \]
\[ a_{mn} = n^{1/2} \delta_{m,n-1}, \quad a^\dagger_{mn} = (n + 1)^{1/2} \delta_{m,n+1}, \]
\[ \hat{q} = (q_{mn}), \quad q_{mn} = \frac{1}{\sqrt{2}} (a^\dagger_{mn} + a_{mn}), \]
\[ \hat{p} = (p_{mn}), \quad q_{mn} = \frac{i}{\sqrt{2}} (a^\dagger_{mn} - a_{mn}). \] (3.55)

Eqs. (3.49), (3.51) and (3.53) show why this representation is called particle number representation. In physics, $a$ is called a destruction operator, while $a^\dagger$ is called a creation operator.

This example shows that different choices of the representation can make the theory superficially different, but physical results like eigenvalues are identical.

### 3.2.2 Path Integral Quantization

The way of using path integral quantization to construct quantum theory is based on the Lagrangian formulation of classical theory. In classical physics, when a particle moves from the initial position $q_1$ at time $t_1$, to a final position $q_2$ at time $t_2$, the Euler-Lagrangian equation (or equivalently the least action principle) only chooses one trajectory. In quantum theory, the particle can take any path to evolve from an initial state $|\Psi(t_1)\rangle$ at time $t_1$ to a final state $|\Psi(t_2)\rangle$ at time $t_2$. Each path has equal probability. Thus, the inner product $\langle \Psi(t_2)|\Psi(t_1)\rangle$, which in physics is called the transition probability amplitude for a quantum system from initial state $|\Psi(t_1)\rangle$ to evolve into the state $|\Psi(t_1)\rangle$, should be given by summing up all the possible paths. In semi-classical approximation, we can think of the path determined by the Euler-Lagrangian equation as the most probable one, while other paths are quantum fluctuations around classical trajectory.

1. Minkowskian Path Integrals in Quantum Mechanics
Starting with the Schrödinger equation (3.17), because $\hat{H}$ is time-independent it has the formal solution,

$$|\Psi(t)\rangle = \exp \left[ -i \int_{t_0}^{t} dt \hat{H} \right] |\Psi(t_0)\rangle = \exp \left[ -i \hat{H} (t - t_0) \right] |\Psi(t_0)\rangle \equiv U(t, t_0)|\Psi(t_0)\rangle. \quad (3.56)$$

In physics, $U(t, t_0)$ is called the evolution operator. Once $U(t, t_0)$ is evaluated, all physical states $\Psi(t)$ at each time $t$ can be determined. In coordinate representation, the state vector $|\Psi(t)\rangle$ is represented by the wave function $\Psi(q, t)$ defined by

$$\Psi(q, t) = \langle q | \Psi(t) \rangle = \int dq_0 \langle q | U(t, t_0) | q_0 \rangle \langle q_0 | \Psi(t_0) \rangle = \int dq_0 \langle q | U(t, t_0) | q_0 \rangle \Psi(q_0, t_0). \quad (3.57)$$

The main task in path integral quantization is to calculate [21]

$$\langle q | U(t, t_0) | q_0 \rangle = \langle q | e^{-i \hat{H}(t-t_0)} | q_0 \rangle. \quad (3.58)$$

The calculation procedure is as follows: First, divide the time-interval $[t, t_0]$ into $n$ small intervals,

$$\Delta t = \frac{t - t_0}{n},$$

and define

$$t_k = t_0 + k \Delta t, \quad k = 0, 1, \cdots, n, \quad t_n \equiv t.$$ 

Then we have

$$\exp \left[ -i \hat{H} (t - t_0) \right] = \exp \left\{ -i \hat{H} \left[ (t_n - t_{n-1}) + \cdots + (t_1 - t_0) \right] \right\} = \prod_{k=1}^{n} \exp \left[ -i \hat{H} (t_k - t_{k-1}) \right]$$

and

$$\langle q | e^{-i \hat{H}(t-t_0)} | q_0 \rangle = \lim_{n \to \infty} \int dq_{n-1} \cdots dq_1 \langle q | q_{n-1} \rangle \langle q_{n-1} | \exp [-i \hat{H} \Delta t] | q_{n-2} \rangle \cdots \langle q_1 | \exp [-i \hat{H} \Delta t] | q_0 \rangle$$

$$= \int \prod_{j=1}^{n-1} dq_j \prod_{k=1}^{n} \langle q_k | \exp \left[ -i \hat{H} \Delta t \right] | q_{k-1} \rangle, \quad (3.59)$$

where $q_n \equiv q$. In Eq. (3.59) we have used (3.20), the completeness of the basis \{ |q\rangle \} of the Hilbert space.

Because $\Delta t$ is small, we consider the approximation to the first order of $\Delta t$,

$$\langle q_k | \exp \left[ -i \hat{H} \Delta t \right] | q_{k-1} \rangle = \langle q_k | q_{k-1} \rangle - i \Delta t \langle q_k | \hat{H} | q_{k-1} \rangle + \mathcal{O}(\Delta t^2)$$

$$= \delta(q_k - q_{k-1}) - i \Delta t \langle q_k | \hat{H} | q_{k-1} \rangle + \mathcal{O}(\Delta t^2), \quad (3.60)$$

where we have used Eq. (3.27).

For later use, consider only the Hamiltonian of the following form for commutative variables,

$$\hat{H} = \frac{1}{2} \hat{p}^2 + V(\hat{q}). \quad (3.61)$$
Hence we have

\[
\langle q_k | \hat{H} | q_{k-1} \rangle = \frac{1}{2} \langle q_k | \hat{p}^2 | q_{k-1} \rangle + \langle q_k | V(q) | q_{k-1} \rangle \\
= \left( -\frac{1}{2} \frac{\partial^2}{\partial q_k^2} + V(q_k) \right) \langle q_k | q_{k-1} \rangle = \left( -\frac{1}{2} \frac{\partial^2}{\partial q_k^2} + V(q_k) \right) \delta(q_k - q_{k-1}).
\] (3.62)

Substituting (3.62) into (3.60) and making use of (3.32), we obtain

\[
\langle q_k | \exp \left[ -i \hat{H} \Delta t \right] | q_{k-1} \rangle = \left[ 1 - i \Delta t \left( -\frac{1}{2} \frac{\partial^2}{\partial q_k^2} + V(q_k) \right) \right] \langle q_k | q_{k-1} \rangle \\
= \int \frac{dp_k}{2\pi} \exp[ip_k(q_k - q_{k-1})] \left\{ 1 - i \Delta t \left[ \frac{1}{2} \hat{p}_k^2 + V(q_k) \right] \right\} \\
= \int \frac{dp_k}{2\pi} \exp[ip_k(q_k - q_{k-1})] \left[ 1 - i \Delta t H(p_k, q_k) \right] \\
\approx \int \frac{dp_k}{2\pi} \exp[ip_k(q_k - q_{k-1}) - i \Delta t H(p_k, q_k)] \\
= \int \frac{dp_k}{2\pi} \exp \left\{ -i \left[ \frac{1}{2} \Delta t \hat{p}_k^2 - p_k(q_k - q_{k-1}) + \Delta t V(q_k) \right] \right\} \\
= \int \frac{dp_k}{2\pi} \exp \left\{ -i \frac{\Delta t}{2} \left[ p_k - \frac{q_k - q_{k-1}}{\Delta t} \right]^2 + i \Delta t \left[ \left( \frac{q_k - q_{k-1}}{\Delta t} \right)^2 - V(q_k) \right] \right\} \\
= \frac{1}{\sqrt{2\pi i \Delta t}} \exp \left\{ i \Delta t \left[ \frac{1}{2} \left( \frac{q_k - q_{k-1}}{\Delta t} \right)^2 - V(q_k) \right] \right\}. \tag{3.63}
\]

In deriving Eq. (3.63), we have used the Gaussian integration formula (3.76).

Substituting (3.63) into (3.59), we obtain

\[
\langle q | e^{-i \hat{H} (t-t_0)} | q_0 \rangle \\
= \lim_{n \to \infty} \int \frac{1}{\sqrt{2\pi i \Delta t}} \prod_{k=1}^{n-1} \frac{dq_k}{\sqrt{2\pi i \Delta t}} \exp \left\{ i \left( \sum_{k=1}^{n} \Delta t \left[ \frac{1}{2} \left( \frac{q_k - q_{k-1}}{\Delta t} \right)^2 - V(q_k) \right] \right) \right\} \\
= \int_{q(t_0)=q_0}^{q(t_0)=q} [dq] \exp \left[ i \int_{t_0}^{t} dt \left( \frac{1}{2} \dot{q}^2 - V(q) \right) \right] \\
= \int_{q(t_0)=q_0}^{q(t_0)=q} [dq] \exp \left[ i \int_{t_0}^{t} dt L(q, \dot{q}) \right] = \int_{q(t_0)=q_0}^{q(t_0)=q} [dq] e^{iS}, \tag{3.64}
\]

where

\[
[dq] \equiv \lim_{n \to \infty} \frac{1}{\sqrt{2\pi i \Delta t}} \prod_{k=1}^{n-1} \frac{dq_k}{\sqrt{2\pi i \Delta t}}
\]

is called the path integral measure.

In a similar way, for a mechanical system described by the Grassman variable \( \chi \) with the following Lagrangian [21],

\[
L = i\dot{\chi}^* - V(\chi^*, \chi) = \frac{i}{2} (\dot{\chi}^* \chi - \dot{\chi} \chi^*) - V(\chi^*, \chi),
\] (3.66)
we can derive the path integral formula,
\[
\langle \chi^* | e^{-i\hat{H}(t-t_0)} | \chi \rangle = \int_{\chi(t_0) = \chi_0}^{\chi(t) = \chi^*} [d\chi^* d\chi] \exp \left[ i \int dt L \right] = \int_{\chi(t_0) = \chi_0}^{\chi(t) = \chi^*} [d\chi^* d\chi] \exp [iS], \tag{3.67}
\]
where the path integral measure for Grassmann variables is defined by
\[
[d\chi^* d\chi] = \lim_{n \to \infty} \prod_{k=1}^{n-1} d\chi_k d\chi_k^*, \quad \chi_k = \chi(t_k), \quad \chi_k^* = \chi^*(t_k),
\]
\[
t_k = t_0 + k\Delta t, \quad \Delta t = \frac{t - t_0}{n}, \quad k = 0, 1, \ldots, n, \tag{3.68}
\]
and especially, \( H = V(\chi, \chi^*) \).

2. Euclidean Path Integrals in Quantum Mechanics

Later in Chapter 4 we shall evaluate the trace (3.124) in supersymmetric quantum mechanics,
\[
\text{Tr} \left[ (-1)^F \exp(-\beta \hat{H}) \right]
\]
where \( \beta \) is a positive number, \( \hat{H} \) is the Hamiltonian of the system, \( F \) is the so-called fermion number, \( F = 0 \mod 2 \) on commutative quantum states, and \( F = 1 \mod 2 \) on anticommutative quantum states. Therefore, we have
\[
\text{Tr} \left[ (-1)^F \exp(-\beta \hat{H}) \right] = \sum_n \langle \psi_n | \left[ (-1)^F \exp(-\beta \hat{H}) \right] | \psi_n \rangle. \tag{3.69}
\]

We can consider \( \beta = \beta - 0 \) and then divide \([\beta, 0]\) into small intervals to use the path integral to calculate the trace. However, there are two distinct features from Eqs. (3.64) and (3.67). First, the trace is a summation taken over diagonal matrix elements, so the initial state and final state vectors should be the same. Furthermore, there is the presence of \((-1)^F\). So the functional integral over commutative variables should be performed with periodic boundary conditions \( \text{(P.B.C.)} \),
\[
q(\beta) = q(0),
\]
while over the Grassmann variable with anti-periodic boundary conditions \( \text{(A.B.C.)} \),
\[
\chi(\beta) = -\chi(0), \quad \text{and} \quad \chi^*(\beta) = -\chi^*(0).
\]
Second, the evolution operator is \( \exp(-\beta \hat{H}) \) rather than \( \exp(-i\beta \hat{H}) \). Thus, we should make the replacement \( t \to -it = \exp(-i\pi/2) t \) in Eqs. (3.64) and (3.67) to evaluate the above trace. In physics, this operation is called \textit{Wick rotation}. Therefore, in a supersymmetric quantum mechanical system composed of both commutative and Grassmann variables, we have \([17, 21]\)
\[
\text{Tr} \left[ (-1)^F \exp(-\beta \hat{H}) \right] = \int_{\text{P.B.C.}} [dq] \int_{\text{A.B.C.}} [d\chi^* d\chi] \exp (-S_E). \tag{3.70}
\]
In (3.70), \( S_E \) denotes the Euclidean action,
\[
S_E = \int dt L_E, \quad L_E = -L[q, i\dot{q}; \chi, i\dot{\chi}, \chi^*, i\dot{\chi}^*]. \tag{3.71}
\]
3. Calculating Path Integral with Semi-classical Approximation

Most of the functional integrals cannot be calculated exactly except a special case called Gaussian integrals in which the Lagrangian is at most quadratic in variables. In the case that the interaction in the physical system is very weak, we can use semi-classical approximation to evaluate the functional integration. The basic idea is to first find a solution to the classical equation of motion, i.e., a classical trajectory, and to expand the theory around the classical solution. Then calculate this solution using the Gaussian integral, and use the perturbation method to deal with higher order terms. That is, consider a classical physical system represented by a classical (Euclidean) action

\[ S[\phi], \ \phi = \{\phi_i\} \]

representing either a set of real commutative, or a set of Grassmann variables. Suppose that \( \phi_{cl} \) is the solution to the classical equation of motion,

\[ \frac{\delta S[\phi]}{\delta \phi_i} = \frac{\partial L}{\partial \phi_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}_i} = 0. \]

Then we define

\[ \phi = \phi_{cl} + \varphi, \quad (3.72) \]

and expand \( S[\phi] \),

\[ S[\phi] = S[\phi_{cl} + \varphi] \]

\[ = S[\phi_{cl}] + \frac{\delta S}{\delta \phi_i} \bigg|_{\phi = \phi_{cl}} \varphi + \frac{1}{2} \int dt \frac{\delta^2 S}{\delta \phi_i \delta \phi_j} \bigg|_{\phi = \phi_{cl}} \varphi_i(t) \varphi_j(t) + O(\varphi^3) \]

\[ = S[\phi_{cl}] + \frac{1}{2} \int dt \varphi_i a_{ij} \varphi_j + O(\varphi^3), \quad (3.73) \]

where \( a_{ij} \equiv \frac{\delta^2 S/(\delta \phi_j \delta \phi_j)}{\phi = \phi_{cl}} \). Hence up to the quadratic term, we obtain

\[ \int [d\phi] \exp (-S[\phi]) \approx \exp (-S[\phi_{cl}]) \int [d \varphi] \exp \left[ -\frac{1}{2} \int dt \varphi_i a_{ij} \varphi_j \right]. \quad (3.74) \]

Using the formula (3.82) for real commutative variables, or equation (B.29) for real Grassmann variables, we have

\[ \int [d\varphi] \exp \left[ -\frac{1}{2} \int dt \varphi_i a_{ij} \varphi_j \right] = \begin{cases} \text{(det } a)^{-1/2}, & \varphi = \text{ real commutative variables;} \\ \pm (\text{det } a)^{1/2}, & \varphi = \text{ real Grassmann variables.} \end{cases} \quad (3.75) \]

4. Gauss Functional Integration Formulas

Finally we display some Gaussian integral formulas for commutative variables which have been used earlier, or will be used later, and the counterparts for Grassmann variables are shown in Appendix B. First, consider the following formulas from Calculus,

\[ \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) = 1, \quad \text{and} \]

\[ \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \exp \left( -ax^2 + bx \right) = \sqrt{\frac{\Gamma}{2a}} \exp \left( -\frac{b^2}{4a} \right), \quad (3.76) \]

where \( a \) and \( b \) can be either real or complex numbers, and \( \text{Re}(a) > 0 \).
To evaluate the Gaussian functional integration,
\[
\int \left[ \frac{d\phi(t)}{\sqrt{2\pi}} \right] \exp \left[ -\frac{1}{2} \int_{-\infty}^{\infty} dt \, \phi^2(t) \right],
\]
we usually use the following approach: first assume \( t \in [-T/2, T/2] \) and impose the periodic boundary condition (P.B.C.),
\[
\phi[-T/2] = \phi[T/2],
\]
and take the limit \( T \to \infty \) in the final stage. The P.B.C. implies that \( \phi(t) \) admits the following Fourier expansion,
\[
\phi(x) = \sum_{k=\infty}^{\infty} \frac{1}{\sqrt{T}} \phi_k \exp \left( \frac{2\pi k t}{T} \right), \quad k \in \mathbb{Z}.
\]

For convenience of discussion, choose \( \phi_0 = 0 \). Later we shall see the case when \( \phi_k \neq 0 \) is highly nontrivial. Hence we have
\[
\int \left[ d\phi(t) \right] \exp \left[ -\frac{1}{2} \int_{-\infty}^{\infty} dt \, \phi^2(t) \right] = \int \prod_{k \neq 0} \frac{d\phi_k}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \sum_{k \neq 0} \phi_k \phi_{-k} \right]
= \int \prod_{k>0} \frac{d\phi_k}{\sqrt{2\pi}} \frac{d\phi_{-k}}{\sqrt{2\pi}} \exp \left( \sum_{k>0} \phi_k \phi_{-k} \right)
= \prod_{k>0} \int \frac{d\phi_k}{\sqrt{2\pi}} \frac{d\phi_{-k}}{\sqrt{2\pi}} \exp \left( \phi_k \phi_{-k} \right)
= 1.
\]

The orthogonal relations
\[
\int_{-T/2}^{T/2} \exp \left( \frac{2\pi k t}{T} \right) \exp \left( \frac{2\pi l t}{T} \right) = T \delta_{k,-l},
\]
and the definition (3.65) of path integral measure, along with the integration formula
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy \sqrt{2\pi} \sqrt{2\pi} \exp(-xy)
= i \int_{-\infty}^{\infty} du \sqrt{2\pi} \exp \left( -\frac{1}{2} u^2 \right) \int_{-\infty}^{\infty} dv \sqrt{2\pi} \exp \left( \frac{1}{2} v^2 \right) = 1,
\]
\[
u = \frac{1}{\sqrt{2}} (x + y), \quad u = \frac{1}{\sqrt{2}} (x - y).
\]

were used in deriving (3.79). From (3.79) we can find the Gaussian functional integration over a set of real commutative variables \( \{\phi^i\} \),
\[
\int [d\phi] \exp \left[ -\frac{1}{2} \int dt \, \phi_i a_{ij} \phi_j \right] = \frac{1}{\sqrt{\det(a_{ij})}}.
\]

Further, the Gaussian functional integration over a set of complex conjugate commutative variables can be obtained from (3.82) by replacing the complex conjugate pair by two real variables,
\[
\int [d\phi^* d\phi] \exp \left[ -\int dt \, \phi_i^* a_{ij} \phi_j \right] = \frac{1}{\det(a_{ij})}.
\]
3.3 Supersymmetric Quantum Mechanics

Supersymmetric quantum mechanics describes particles with half-integer spins moving in a particular potential well. It was originally proposed by Witten as toy models to study dynamical supersymmetry breaking [6, 7, 13]. It turned out that during the past 30 years, these types of quantum mechanical models, especially a special class of them called 0+1-dimensional supersymmetric non-linear models, have presented rich mathematical structures. Two of the remarkable branches in differential topology, the Atiyah-Singer index theorem and Morse inequalities, can be derived from 0+1-dimensional supersymmetric non-linear models [7, 15, 16, 17, 22].

3.3.1 Supersymmetry Algebra of Supersymmetric Quantum Mechanical System

The construction of a supersymmetric quantum mechanical system is dominated by the following graded algebra [6],

\[
[Q_i, H] = 0, \quad \{Q_i, Q_j\} = 2H\delta_{ij}, \quad i, j = 1, \ldots, 2N,
\]

where \(H\) is the Hamiltonian of the system, and the \(Q_i\) are generators of supersymmetry. At the classical level, \(Q_i\) can be obtained by the Nöther theorem from the invariance of classical action under a supersymmetry transformation. At the quantum level, the \(Q_i\) are operators in the Hilbert space of the system.

The supersymmetry algebra (3.84) means

\[
Q_i^2 = H, \quad \{Q_i, Q_j\} = 0 \quad \text{for} \quad i \neq j.
\]

In the case when \(N = 1\), it is convenient to define [7]

\[
Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2), \quad \text{and} \quad Q^* = \frac{1}{\sqrt{2}}(Q_1 - iQ_2).
\]

Consequently, the \(N=1\) supersymmetry algebra takes the following form,

\[
Q^2 = Q^*2 = 0, \quad \{Q, Q^*\} = H, \quad \text{and} \quad [H, Q] = [H, Q^*] = 0.
\]

3.3.2 An Example of \(N=1\) supersymmetric Quantum Mechanics

Consider the following example using \(N = 1\) supersymmetry algebra to reveal the structure of the Hilbert space for an \(N = 1\) supersymmetric quantum mechanics. The model is described by the following Lagrangian [6, 23, 24],

\[
L = \frac{1}{2} \dot{x}^2 + i \psi^* \dot{\psi} - \frac{1}{2} W^2(x) - \frac{1}{2} [\psi^*, \psi] W'(x),
\]

where \(W(x)\) is an arbitrary function of \(x\) and,

\[
x = x(t), \quad \dot{x} = \frac{dx}{dt}, \quad W'(x) = \frac{\partial W}{\partial x}.
\]
with Grassman variables $\psi(t)$ and $\psi^*(t)$ (see Appendix B),

$$\dot{\psi} = \frac{dL}{dt}, \quad \{\psi, \psi\} = \{\psi^*, \psi^*\} = 0. \quad (3.90)$$

The classical action

$$S = \int dt \, L \left[ x, \dot{x}, \psi, \dot{\psi} \right] \quad (3.91)$$

is invariant under the following supersymmetric transformations,

$$\begin{align*}
\delta x &= i(\psi \epsilon + \epsilon^* \psi^*), \\
\delta \psi &= \epsilon^* \dot{x} - i \epsilon W, \\
\delta \psi^* &= \epsilon \dot{x} + i \epsilon W, 
\end{align*} \quad (3.92)$$

since

$$\delta S = \int dt \, \delta L = \int dt \, \frac{dL}{dt} \left[ -i \epsilon \psi (\dot{x} - iW) + \frac{1}{2} \epsilon^* \psi^* (\dot{x} + iW) \right] = 0. \quad (3.93)$$

In (3.92) $\epsilon$ and $\epsilon^*$ are constant Grassmann parameters of the transformation.

By means of the Nöther theorem introduced in Appendix A, we find the generators of supersymmetry transformations (up to a normalization constant factor) to be,

$$Q = \frac{1}{\sqrt{2}} (\dot{x} + iW) \psi^*, \quad \text{and} \quad Q^* = \frac{1}{\sqrt{2}} (\dot{x} - iW) \psi. \quad (3.94)$$

To find the Hamiltonian of the model, according to the procedure stated in Sect. 3.1, choose the coordinates

$$q^\alpha = (x, \psi). \quad (3.95)$$

The conjugate momenta are

$$p^\alpha = (p, \pi), \quad p = \frac{\partial L}{\partial \dot{x}} = \dot{x}, \quad \text{and} \quad \pi = \frac{\partial L}{\partial \dot{\psi}} = i \psi^*. \quad (3.96)$$

The Hamiltonian is given by the Legendre transformation,

$$H (x, \psi, p, \psi^*) = \frac{1}{2} \left[ p^2 + W^2(x) \right] + \frac{1}{2} [\psi^*, \psi] W'(x). \quad (3.97)$$

According to the procedure of canonical quantization shown in Sect. 3.2, all the quantities should be replaced by operators, and the following canonical commutation and anticommutation relations should be imposed,

$$\begin{align*}
[\hat{x}, \hat{p}] &= i, \\
[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] &= 0, \quad \left\{ \hat{\psi^*}, \hat{\psi} \right\} = 1, \\
\left\{ \hat{\psi}, \hat{\psi} \right\} &= \left\{ \hat{\psi^*}, \hat{\psi^*} \right\} = 0. \quad (3.98)\end{align*}$$

To construct the Hilbert space of the quantum mechanical system, we should choose representations for operators. For the commutative operators $\hat{x}$ and $\hat{p}$, the usual coordinate representation,

$$\hat{x} = x, \quad \text{and} \quad \hat{p} = -i \frac{\partial}{\partial x}. \quad (3.99)$$
is chosen. For the anticommutative operators, it is convenient to use the “particle number” representation. We can choose

\[ \hat{\psi} = \frac{1}{2}(\sigma_1 - i\sigma_2) \equiv \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \hat{\psi}^* = \frac{1}{2}(\sigma_1 + i\sigma_2) \equiv \sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (3.100) \]

Further, there exists

\[ [\hat{\psi}^*, \hat{\psi}] = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (3.101) \]

where \( \sigma_i, i = 1, 2, 3, \) are the Pauli matrices [12]

\[ \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (3.102) \]

It can easily be checked that the representations (3.99) and (3.100) indeed satisfy the operator relations (3.98).

Hence, the supercharge and the Hamiltonian operators take the forms

\[ Q = \frac{1}{\sqrt{2}} (p + iW) \psi^* = \frac{1}{\sqrt{2}} \left( \frac{-i}{\partial x} + iW \right) \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + W^2(x) \right) \sigma_+ = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}, \]

\[ Q^* = \frac{1}{\sqrt{2}} (p - iW) \psi = \frac{1}{\sqrt{2}} \left( \frac{-i}{\partial x} - iW \right) \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - W^2(x) \right) \sigma_- = \begin{bmatrix} 0 & 0 \\ D^* & 0 \end{bmatrix}, \]

\[ H = Q^*Q + QQ^* = \frac{1}{2} \left( \frac{-\partial^2}{\partial x^2} + W^2(x) \right) + \frac{1}{2} W'(x) \sigma_3 = \begin{bmatrix} DD^* & 0 \\ 0 & D^*D \end{bmatrix} = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}, \quad (3.103) \]

where

\[ D = \frac{1}{\sqrt{2}} \left( \frac{-i}{\partial x} + iW \right), \quad D^* = \frac{1}{\sqrt{2}} \left( \frac{-i}{\partial x} - iW \right), \]

\[ H_+ = DD^* = \frac{1}{2} \left( \frac{-\partial^2}{\partial x^2} + W^2(x) + W'(x) \right), \]

\[ H_- = D^*D = \frac{1}{2} \left( \frac{-\partial^2}{\partial x^2} - W^2(x) + W'(x) \right). \quad (3.104) \]

Then it can be verified by a straightforward calculation that the \( N = 1 \) superalgebra (3.87) is indeed realized in this model.

In the following, observe the structure of the Hilbert space of this supersymmetric quantum mechanical model. First, according to one of the fundamental principles of quantum mechanics [11], a quantum state is specified by a common eigenstate of a complete set of commutative Hermitian operators. For the model at hand, the Hamiltonian \( H \) and \( \psi^* \psi = \sigma_3 \) constitute a complete set since there exists

\[ [H, \psi^* \psi] = [H, \sigma_3] = 0. \quad (3.105) \]
Because $\sigma_3$ has two linearly independent eigenvectors,

$$\sigma_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \sigma_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

(3.106)

the Hilbert space $\mathcal{H}$ of the model is the eigenspace spanned by the two eigenvectors. Hence a quantum state must be of the form of two-component spinor wave functions,

$$\Psi(x) = \langle x | \Psi \rangle = \Psi_+(x) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \Psi_-(x) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \Psi_+(x) \\ \Psi_-(x) \end{bmatrix},$$

(3.107)

where $\Psi_+(x)$ and $\Psi_-(x)$ can further be determined by the eigenvalue equation of Hamiltonian operators

$$H \Psi(x) = E \Psi(x),$$

(3.108)

i.e.,

$$\begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} \begin{bmatrix} \Psi_+(x) \\ \Psi_-(x) \end{bmatrix} = E \begin{bmatrix} \Psi_+(x) \\ \Psi_-(x) \end{bmatrix},$$

$$H_+ \Psi_+(x) = E_+ \Psi_+(x), \quad H_- \Psi_-(x) = E_- \Psi_-(x).$$

(3.109)

From Eq. (3.85), $H = Q_1^2 = Q_2^3$, hence there must exist

$$E \geq 0.$$

(3.110)

Eqs. (3.106), (3.107) and (3.109) imply that the Hilbert space $\mathcal{H}$ can be decomposed into a direct sum of two subspaces graded by the eigenvalues of $\sigma_3$,

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F = \begin{bmatrix} \mathcal{H}_B \\ \mathcal{H}_F \end{bmatrix},$$

$$\mathcal{H}_B = \{ | \Psi_B \rangle = \begin{bmatrix} \Psi_+(x) \\ 0 \end{bmatrix} \}, \quad \mathcal{H}_F = \{ | \Psi_F \rangle = \begin{bmatrix} 0 \\ \Psi_-(x) \end{bmatrix} \}$$

$$\sigma_3 | \Psi_B \rangle = | \Psi_B \rangle, \quad \sigma_3 | \Psi_F \rangle = -| \Psi_F \rangle.$$

(3.111)

In physics terminology, $\mathcal{H}_B$ and $\mathcal{H}_F$ are called the “bosonic” and “fermionic” sectors of the Hilbert space, respectively.

Using the explicit representations of $Q^*$, $Q$ and $H$ listed in Eq. (3.103), we have

$$Q^* \Psi_B(x) = \begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix} \begin{bmatrix} \Psi_+(x) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ D^* \Psi_+(x) \end{bmatrix};$$

$$Q \Psi_F(x) = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \Psi_-(x) \end{bmatrix} = \begin{bmatrix} D \Psi_-(x) \\ 0 \end{bmatrix},$$

$$H \Psi_B(x) = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} \begin{bmatrix} \Psi_+(x) \\ 0 \end{bmatrix} = \begin{bmatrix} H_+ \Psi_+(x) \\ 0 \end{bmatrix};$$

$$H \Psi_F(x) = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} \begin{bmatrix} 0 \\ \Psi_+(x) \end{bmatrix} = \begin{bmatrix} 0 \\ H_- \Psi_+(x) \end{bmatrix}. $$

(3.112)
These results imply the following mappings between $\mathcal{H}_B$ and $\mathcal{H}_F$,

\begin{align*}
Q^* : \mathcal{H}_B & \rightarrow \mathcal{H}_F, \\
Q : \mathcal{H}_F & \rightarrow \mathcal{H}_B, \\
H : \mathcal{H}_B & \rightarrow \mathcal{H}_B, \\
\mathcal{H}_F & \rightarrow \mathcal{H}_F.
\end{align*}

(3.113)

In physics, $\sigma_3$ is usually expressed in terms of a so-called “fermion number” operator $F$ [7, 13],

\begin{align*}
\sigma_3 &= (-1)^F, \\
F &= \frac{1}{2} (1 - \psi^* \psi) = \frac{1}{2} (1 - \sigma_3) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
F|\Psi_B\rangle &= 0, \quad F|\Psi_F\rangle = |\Psi_F\rangle, \\
(-1)^F|\Psi_B\rangle &= \Psi_B, \quad (-1)^F|\Psi_F\rangle = -|\Psi_F\rangle.
\end{align*}

(3.114)

We can easily show from Eqs. (3.103) and (3.114) that

\[ [H, (-1)^F] = 0, \quad (-1)^F Q + Q (-1)^F = 0, \quad \text{and} \quad (-1)^F Q^* + Q^* (-1)^F = 0. \]

(3.115)

### 3.3.3 General Graded Structure of the Hilbert Space of Supersymmetric Quantum Mechanics and Witten Index

The features of a Hilbert space shown in the above example are universal for any supersymmetric quantum mechanical system. Due to the underlain supersymmetry algebra, the Hilbert space $\mathcal{H}$ of a supersymmetric quantum physical system is always decomposed into the direct sum of two graded subspaces,

\[ \mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F, \]

(3.116)

where $\mathcal{H}_B$ and $\mathcal{H}_F$ are called the spaces of “bosonic” and “fermionic” states, respectively. A supersymmetric quantum theory is defined as a quantum theory with symmetry operators $Q_i$, $i = 1, \cdots, 2N$ in the Hilbert space, which can map $\mathcal{H}_B$ into $\mathcal{H}_F$ and vice-versa,

\[ Q_i : \mathcal{H}_B \rightarrow \mathcal{H}_F \quad \text{or} \quad Q_i : \mathcal{H}_F \rightarrow \mathcal{H}_B. \]

(3.117)

Further, the two subspaces are distinguished by the operator $(-1)^F$, which is defined as follows [7, 22]:

\[ (-1)^F |\lambda\rangle = |\lambda\rangle \quad \text{for} \quad |\lambda\rangle \in \mathcal{H}_B, \quad \text{or} \quad (-1)^F |\chi\rangle = -|\chi\rangle \quad \text{for} \quad |\chi\rangle \in \mathcal{H}_F. \]

(3.118)

Then we have following two results:

**Theorem 3.3.1** There exist the following commutation and anticommutation relations

\[ [H, (-1)^F] = 0, \quad \text{and} \quad (-1)^F Q_i + Q_i (-1)^F = 0. \]

(3.119)
This result can be easily proved using supersymmetry algebra (3.85), supersymmetry mapping (3.117), and the definition (3.118) of $(-1)^F$.

**Theorem 3.3.2** The eigenvectors with non-zero eigenvalues of the Hamiltonian operator $H$ always arise in pairs from the bosonic sector $\mathcal{H}_B$ and the fermionic sector $\mathcal{H}_F$ [7, 15].

**Proof:** Let $|\lambda\rangle \in \mathcal{H}_B$, and let $H|\lambda\rangle = E|\lambda\rangle$ and $E \neq 0$. From Eq. (3.85), $H = Q_i^2$, so $E > 0$. Then define a state vector

$$|\chi\rangle = \frac{1}{\sqrt{E}} Q_i |\lambda\rangle.$$  (3.120)

Obviously, $|\chi\rangle \in \mathcal{H}_F$ since $Q$ maps a bosonic state into a fermionic state. Using $[H, Q_i] = 0$, we have

$$H|\chi\rangle = \frac{1}{\sqrt{E}} Q_i H|\lambda\rangle = E|\chi\rangle.$$  (3.121)

So both $|\chi\rangle \in \mathcal{H}_F$ and $|\lambda\rangle \in \mathcal{H}_B$ are the eigenvectors of $H$ with the same eigenvalue. Further, from $H = Q_i^2$, there exists

$$Q_i |\lambda\rangle = \sqrt{E}|\chi\rangle, \quad \text{and} \quad Q_i |\chi\rangle = \sqrt{E}|\lambda\rangle.$$  (3.122)

Therefore, all the state vectors in the Hilbert space $\mathcal{H}$ corresponding to non-zero eigenvalues of the Hamiltonian operator appear in boson-fermion pairs, furnishing a two-dimensional representation of supersymmetry algebra (3.84) or (3.85) for each value $E$ allowed by physics.

Theorem 3.2.2 works only for those state vectors corresponding to non-zero eigenvalues of $H$, while the states corresponding to $E = 0$ may not be paired. Since $Q_i^2 = H$, if $H|\lambda\rangle = 0$ for $|\lambda\rangle \in \mathcal{H}_B$ (or $H|\psi\rangle = 0$ for $|\psi\rangle \in \mathcal{H}_F$), then $Q_i |\lambda\rangle = 0$ (or $H|\psi\rangle = 0$). Hence, these state vectors form trivial one-dimensional representations of the supersymmetry algebra. As Witten pointed out [7], in general, depending on a concrete physical model, there may be an arbitrary number $n_B^{E=0}$ of bosonic state vectors with $E = 0$, and an arbitrary number $n_F^{E=0}$ of fermionic state vectors with $E = 0$. When the parameters in a physical model, such as masses of particles, coupling constants of particle interaction, and the volume of space vary, the numbers $n_B^{E=0}$ and $n_F^{E=0}$ may change since some state vectors corresponding to $E \neq 0$ may become vectors with $E = 0$ and vice-versa. However, as long as supersymmetry does not break, according to Theorem 3.3.2, the state vectors in $\mathcal{H}$ with $E \neq 0$ always arise in pairs, so the difference $n_B^{E=0} - n_F^{E=0}$ always stays constant. Therefore, there exists the following result introduced by Witten [7], which is a direct consequence of Theorem 3.2.2:

**Theorem 3.3.3** Let $n_B^{E=0}$ and $n_F^{E=0}$ be the numbers of bosonic and fermionic eigenvectors of the Hamiltonian operators corresponding to $E = 0$ in the Hilbert space $\mathcal{H}$, respectively. Then there exists [7]

$$n_B^{E=0} - n_F^{E=0} = \text{Tr}(-1)^F,$$  (3.123)

where the trace is taken over all the state vectors in the Hilbert space $\mathcal{H}$.

This result is obvious from Theorem 3.2.2 and the definition (3.118) of $(-1)^F$. The state vectors corresponding to $E \neq 0$ do not contribute to $\text{Tr}(-1)^F$ because for every state $|\lambda\rangle \in \mathcal{H}_B$
with $E \neq 0$ that contribute +1 to the trace, there is a state $|\chi\rangle = 1/\sqrt{E}Q_i|\lambda\rangle \in \mathcal{H}_F$ with $E \neq 0$ that contribute −1 and cancel the contribution from $|\lambda\rangle$. Therefore, $\text{Tr}(-1)^F$ can only be contributed to by the state vectors with $E = 0$ and hence equals $n_{E=0}^B - n_{E=0}^F$.

$\text{Tr}(-1)^F$ is nowadays termed as the Witten index. Witten initially introduced this notion mainly for observing dynamical supersymmetry breaking [7]. If $\text{Tr}(-1)^F \neq 0$, there must exist at least one state vector $|\varphi\rangle$ (either in $\mathcal{H}_B$ or $\mathcal{H}_F$) such that $H|\varphi\rangle = 0$, and hence $Q_i|\varphi\rangle = 0$. Therefore, the supersymmetry does not break. In the meantime, he realized that $\text{Tr}(-1)^F$ is the index for the operator of supersymmetry generator. Further, he argued that the 0+1-dimensional supersymmetric sigma model should lead to the Atiyah-Singer index theorem [7].

From a mathematical viewpoint, $\text{Tr}(-1)^F$ is ill defined because the Hilbert space is infinitely dimensional and the infinite summation over all the state vectors is in general not absolutely convergent. Therefore, when calculating $\text{Tr}(-1)^F$ in a supersymmetric physical system, one usually adopts the regularized version as

$$\text{Tr}(-1)^F|_R \equiv \text{Tr} \left[ (-1)^F \exp(-\beta H) \right], \quad (3.124)$$

where $\beta$ is an arbitrary positive parameter, and $H$ is the Hamiltonian. Certainly $\text{Tr}(-1)^F \exp(-\beta H)$ is independent of $\beta$ because the state vectors with $E \neq 0$ do not contribute.

Finally, consider the correspondence between a supersymmetric quantum mechanical system and the de Rham Cohomology and Hodge Theory established on the exterior algebra $\Lambda^*$. The details will be shown in Chapter 4. Comparing the supersymmetry algebra (3.87), the decomposition (3.116) of the Hilbert space, the definition (3.117) of supersymmetry transformation and the definition (3.118) of the grading operator $(-1)^F$ with Eqs.(2.11), (2.46) and (2.50) in Sect. 2.1., we have the following identifications:

$$\mathcal{H} \leftrightarrow \Lambda^* = \{\Lambda^p\},$$
$$\mathcal{H}_B \leftrightarrow \Lambda^e = \bigoplus_{p=2k} \Lambda^p,$$
$$\mathcal{H}_F \leftrightarrow \Lambda^o = \bigoplus_{p=2k+1} \Lambda^p,$$
$$Q \leftrightarrow d,$$
$$Q^* \leftrightarrow \delta,$$
$$H = Q^*Q + QQ^* \leftrightarrow \Delta = d\delta + d\delta,$$
$$(-1)^F \leftrightarrow (-1)^k, \quad k = \text{even, odd}. \quad (3.125)$$
Chapter 4

Derivation of Atiyah-Singer Index Theorem from Supersymmetric Quantum Mechanics

4.1 Introducing 0+1-dimensional Supersymmetric Non-Linear Sigma Model

In physics, the supersymmetric non-linear sigma model in 0+1 dimension describes a certain physical object (e.g., a superstring) with a tower of spins (up to spin \( n/2 \)) moving an \( n \)-dimensional manifold \( M_n \). Here we consider \( M_n \) to be a compact Riemannian manifold without boundary for deriving the Atiyah-Singer index theorem. The Lagrangian of the model is [7, 15, 17]

\[
L = \frac{1}{2} g_{\mu\nu}(x) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + \frac{i}{2} g_{\mu\nu}(x) \overline{\psi}^\mu \gamma^0 \frac{D\psi^\nu}{dt} + \frac{1}{12} R_{\mu\nu\lambda\rho}(x) \overline{\psi}^\mu \psi^\lambda \psi^\nu \psi^\rho, \\
\mu, \nu, \lambda, \rho = 0, 1, \ldots, n - 1. \tag{4.1}
\]

The various quantities appearing in \( L \) are listed as follows: \( x^\mu(t) \) (\( \mu = 0, 1, \ldots, n - 1 \)) are local coordinates on the Riemannian manifold \( M_n \), \( g_{\mu\nu}(x) \) \( (\mu, \nu = 0, 1, \ldots, n) \) is the metric of the manifold, \( R_{\mu\nu\lambda\rho} \) is the curvature of the manifold given in Eq. (2.31). \( \psi_\mu \) is a two-component real spinor with,

\[
\psi^\mu(t) = \begin{bmatrix} \psi^\mu_1(t) \\ \psi^\mu_2(t) \end{bmatrix}, \quad \psi^\mu_r = \psi^{\mu^r}, \quad r = 1, 2; \\
\overline{\psi}^\mu = \psi^{\mu T} \gamma^0, \quad \gamma_0 = \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},
\]

\[
\frac{D\psi^\nu}{dt} = \frac{d\psi^\nu}{dt} + \Gamma^\nu_{\lambda\rho} \frac{dx^\lambda}{dt} \psi^\rho, \tag{4.2}
\]

where \( \Gamma^\nu_{\lambda\rho} \) is the Christoffel symbol shown in (2.30). The action \( S = \int dt L \) is invariant under the following supersymmetry transformations,

\[
\delta x^\mu = \xi \psi^\mu, \\
\delta \psi^\mu = -i \gamma^0 \dot{x}^\mu \epsilon - \Gamma^\nu_{\nu\rho} (\overline{\xi} \psi^\nu) \psi^\rho. \tag{4.3}
\]
Derivation of Index Theorem

Above, $\dot{x}^\nu = dx^\nu/dt$, and $\epsilon$ is a real two-component constant anticommuting spinor,

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}, \quad \tau = \epsilon^T \gamma_0. \quad (4.4)$$

In the following sections, we shall use this model and its variants to derive the Atiyah-Singer index theorems for four classical complexes introduced in Sect. 2.4.

4.2 Gauss-Bonnet Theorem from 0+1-dimensional N=1 Supersymmetric Sigma Model

4.2.1 Identification between Quantum N=1 Supersymmetric Sigma Model and de Rham Complex

To establish the identification between the quantum N=1 supersymmetric sigma model and the de Rham complex, first recombine two real components of the spinor,

$$\chi^\mu = \frac{1}{\sqrt{2}} (\psi^\mu_1 + i \psi^\mu_2), \quad and \quad \chi^{*\mu} = \frac{1}{\sqrt{2}} (\psi^\mu_1 - i \psi^\mu_2). \quad (4.5)$$

Then the original Lagrangian (4.1) can be expressed as,

$$L = \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + ig_{\mu\nu}(x) \chi^{*\mu} D\chi^\nu dt - \frac{1}{4} R_{\mu\nu\lambda\rho} \chi^{*\mu} \chi^{*\nu} \chi^\lambda \chi^\rho. \quad (4.6)$$

The supersymmetry transformations (4.3) become

$$\delta x^\mu = \eta \chi^{*\mu} - \eta^* \chi^\mu, \quad \delta \chi^\mu = \eta \left( i \dot{x}^\mu - \Gamma^\mu_{\nu\rho} \chi^{*\nu} \chi^\rho \right), \quad \text{and} \quad \delta \chi^{*\mu} = \eta^* \left( -i \dot{x}^\mu - \Gamma^\mu_{\nu\rho} \chi^\nu \chi^{*\rho} \right), \quad (4.7)$$

where

$$\eta = \frac{1}{\sqrt{2}} (\epsilon_1 + i \epsilon_2), \quad \text{and} \quad \eta^* = \frac{1}{\sqrt{2}} (\epsilon_1 - i \epsilon_2). \quad (4.8)$$

Using the Nöther theorem, we obtain the generators of supersymmetry transformations,

$$Q = -i \chi^\mu_{\nu} \left( i \dot{x}^\mu - \Gamma^\mu_{\nu\rho} \chi^{*\nu} \chi^\rho \right), \quad \text{and} \quad Q^* = i \chi^\mu_{\nu} \left( i \dot{x}^\mu - \Gamma^\mu_{\nu\rho} \chi^\nu \chi^{*\rho} \right). \quad (4.9)$$

Choose canonical coordinates $(x_\mu, \psi_\mu)$, the corresponding canonical conjugate momenta are

$$p^\mu = \frac{\partial L}{\partial \dot{x}^\mu} = g^{\mu\nu} \left( \dot{x}^\nu + i \Gamma^\nu_{\nu\rho} \chi^{*\nu} \chi^\rho \right), \quad \text{and} \quad \pi^\mu = \frac{\partial L}{\partial \chi^\mu} = ig^{\mu\nu} \chi^{*\nu}. \quad (4.10)$$
Now we use canonical quantization to construct quantum theory. The formal canonical commutation and anticommutation relations are

\[
[x_\mu, p_\nu] = i\delta_\mu^\nu, \quad [x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \\
\{\chi^\mu, \chi^{\ast \nu}\} = g^{\mu \nu}(x), \quad \{\chi^\mu, \chi^\nu\} = \{\chi^{\ast \mu}, \chi^{\ast \nu}\} = 0. \tag{4.11}
\]

Similar to the example shown in Sect. 3.3, choose the coordinate representation for the commutative variable, and the particle number representation for the Grassmann variable. Then \( p_\mu = -i\partial /\partial x^\mu \), \( \chi^{\ast \mu} \) acts as a creation operator of a fermion, and \( \chi^\mu \) acts as an destruction operator of a fermion. Therefore, the Hilbert space of the quantum theory is

\[
\mathcal{H} = \{|\Psi_p\rangle | \Psi_p\rangle = \omega_{\mu_1 \mu_2 \cdots \mu_p}(x) \chi^{\ast \mu_1} \chi^{* \mu_2} \cdots \chi^{* \mu_p}|\Omega\rangle, \quad p = 1, 2, \cdots, n\},
\]

where \(|\Omega\rangle\) is called the fermionic vacuum defined by \([7, 15, 16]\)

\[
\psi^\mu|\Omega\rangle = 0. \tag{4.13}
\]

In physics, \(|\Psi_p\rangle\) is called a quantum state vector representing \( p \) fermions. From Eq. (4.11), \( \chi^{* \mu} \chi^{* \nu} = -\chi^{\ast \nu} \chi^{\ast \mu} \). Therefore, \(|\Psi_p\rangle\) corresponds to a \( p \)-form on the Riemannian manifold \( M_n \),

\[
|\Psi_p\rangle = \omega_{\mu_1 \mu_2 \cdots \mu_p}(x) \chi^{* \mu_1} \chi^{* \mu_2} \cdots \chi^{\ast \mu_p}|\Omega\rangle \quad \longleftrightarrow \quad \omega_p = \alpha_{\mu_1 \mu_2 \cdots \mu_p}(x) dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}. \tag{4.14}
\]

From Eqs. (4.9) and (4.10), the supersymmetry generators are

\[
Q = i\chi^{\ast \mu} p_\mu, \quad \text{and} \quad Q^* = -i\chi^{\ast \mu} p_\mu. \tag{4.15}
\]

It can easily be verified with Eq. (4.11) that

\[
Q^2 = -\chi^{\ast \mu} p_\mu \chi^{* \nu} p_\nu = -\chi^{* \mu} \chi^{* \nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} = 0, \\
Q^*Q = -\chi^{\ast \mu} p_\mu \chi^{* \nu} p_\nu = -\chi^{\ast \mu} \chi^{* \nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} = 0. \tag{4.16}
\]

Further,

\[
Q|\Psi_p\rangle = \left[ \frac{\partial}{\partial x^\mu} \omega_{\mu_1 \mu_2 \cdots \mu_p}(x) \right] \chi^{* \mu_1} \chi^{* \mu_2} \cdots \chi^{* \mu_p}|\Omega\rangle \sim |\Psi_{p+1}\rangle; \\
Q^*|\Psi_p\rangle = \left( -\frac{\partial}{\partial x^\mu} \omega_{\mu_1 \mu_2 \cdots \mu_p}(x) \right) \left\{ \left\{ \chi^\mu, \chi^{* \mu_1} \right\} \chi^{* \mu_2} \cdots \chi^{* \mu_p} - \chi^{* \mu_1} \left\{ \chi^\mu, \chi^{* \mu_2} \right\} \chi^{* \mu_3} \cdots \chi^{* \mu_p} \\
+ \cdots + (-1)^p \chi^{\ast \mu_1} \chi^{\ast \mu_2} \cdots \chi^{\ast \mu_{p-1}} \left\{ \chi^\mu, \chi^{\ast \mu_p} \right\} \right\} |\Omega\rangle \sim |\Psi_{p-1}\rangle. \tag{4.17}
\]

So there arise identifications

\[
Q \leftrightarrow d, \\
Q^* \leftrightarrow \delta, \\
H = QQ^* + Q^*Q \leftrightarrow \Delta p = d\delta + \delta d, \quad \text{and} \\
(-1)^F \leftrightarrow (-1)^p, \tag{4.18}
\]

where \( F \) is the operator of the fermion number. Therefore, we obtain

\[
\operatorname{Tr}(-1)^F \big|_R = \operatorname{Tr} \left[ (-1)^F e^{-\beta H} \right] = \sum_{p=0}^{n} (-1)^p B_p = \chi(M). \tag{4.19}
\]
4.2.2 Derivation of Gauss-Bonnet Theorem

Once we have identified the physical quantity $\text{Tr}(-1)^F \exp(-\beta H)$ as the index of the de Rham complex, the following step is to employ the standard quantum mechanical method to calculate $\text{Tr} \left[ (-1)^F \exp(-\beta H) \right]$, and hence to derive the Gauss-Bonnet formula.

By means of Eq. (3.70) in Sect. 3.2, we have

$$\text{Tr} \left[ (-1)^F \exp(-\beta H) \right] = \int_{\text{P.B.C.}} [dx^\mu(t) d\chi(t) d\chi^*(t)] \exp \left( -\int_0^\beta dt L[x, \chi, \chi^*] \right), \quad (4.20)$$

where $L$ is shown in Eq. (4.6). Because of the periodic boundary conditions,

$$x^\mu(0) = x^\mu(\beta), \quad \chi^\mu(0) = \chi^\mu(\beta), \quad \text{and} \quad \chi^{*\mu}(0) = \chi^{*\mu}(\beta), \quad (4.21)$$

we have the following Fourier series expansions with frequency $2\pi k/\beta$, $k \in \mathbb{Z}$,

$$x^\mu = \sum_{k=-\infty}^{\infty} x^\mu_k \exp \left( i \frac{2k\pi}{\beta} t \right),$$

$$\chi^\mu = \sum_{k=-\infty}^{\infty} \chi^\mu_k \exp \left( i \frac{2k\pi}{\beta} t \right),$$

$$\chi^{*\mu} = \sum_{k=-\infty}^{\infty} \chi^{*\mu}_k \exp \left( -i \frac{2k\pi}{\beta} t \right). \quad (4.22)$$

In (4.22) $\chi^\mu_k$ and $\chi^{*\mu}_k$ are the Grassmann numbers. The term with frequency $2\pi k/\beta$ is usually called a $k$-mode.

With Eq. (4.22), the action $S_E = \int_0^\beta dt L$ becomes a sum of infinite number of modes due to the orthogonality among complex exponential functions (see Eq. (4.23) below). The functional integral (4.20) converts into an infinite number of multiple integrations over $x^\mu_k$, $\chi^\mu_k$, and $\chi^{*\mu}_k$.

Further, as shown above, the index of the de Rham complex is independent of $\beta$, so we can evaluate the functional integral (4.20) in the small-$\beta$ limit. When $\beta$ is small, only the constant (i.e., $k = 0$) modes in $\exp [-S_E]$ make dominant contributions to the integration, while the non-constant modes, which depend on $2\pi/\beta$, are terms with very small values. In physics terminology, $\beta$ plays a role of the coupling parameter of interactions. The constant modes are strongly coupled, while the non-constant modes are weakly coupled. Thus the functional integral splits into an integral over constant bosonic (or commutative) and fermionic (or Grassmann) configurations which are finite dimensional, and an integral over non-constant configurations. The latter can be calculated by means of a perturbation approach, which actually vanishes due to supersymmetry [17].

The concrete procedure begins by using the expansion (4.22) and the orthogonal relation

$$\int_0^\beta dt \exp \left( i \frac{2k\pi}{\beta} t \right) \exp \left( i \frac{2l\pi}{\beta} t \right) = \beta \delta_{k,-l}, \quad (4.23)$$

to obtain

$$S_E = \frac{1}{2} \beta \sum_{k \neq 0} \left( \frac{2k\pi}{\beta} \right)^2 g_{\mu\nu}(x_0) x^\mu_k x^\nu_k - \beta \sum_{k \neq 0} g_{\mu\nu}(x_0) \chi^{*\mu}_k \chi^{\nu}_k \frac{2k\pi}{\beta}$$

$$- \frac{1}{4} \beta R_{\mu\nu\lambda\rho}(x_0) \chi^{*\mu}_0 \chi^{\nu}_0 \chi^{*\lambda}_0 \chi^{\rho}_0 + \mathcal{O}(\beta^2). \quad (4.24)$$
Gauss-Bonnet Theorem

Hence

\[
\text{Tr} \left[ (-1)^F \exp(-\beta H) \right] = \int \left\{ \prod_{\mu=1}^{n} \left[ \frac{dx_\mu^\mu}{(2\pi)^{1/2}} d\chi^\mu_0 d\chi^\mu_0 \right] \exp \left[ -\frac{1}{4} \beta R_{\mu\nu\lambda\rho}(x_0) \chi^\mu_0 \chi^\nu_0 \chi^\lambda_0 \chi^\rho_0 \right] \right\} \\
\times \int \prod_{k \neq 0} dx_k^\mu \exp \left[ \frac{1}{2} \beta g_{\mu\nu}(x_0) \sum_{k \neq 0} \left( \frac{2\pi}{\beta} \right)^2 x_k^\mu x_k^\nu \right] \\
\times \prod_{k \neq 0} \int d\chi_k^\mu d\chi_k^\mu \exp \left[ -\beta g_{\mu\nu}(x_0) \sum_{k \neq 0} \chi^\mu_k \chi^\nu_k \frac{2\pi}{\beta} \right] \\
= \int \prod_{\mu=1}^{n} \left[ \frac{dx_0^\mu}{(2\pi)^{1/2}} d\chi_0^\mu d\chi_0^\mu \right] \exp \left[ -\frac{1}{4} \beta R_{\mu\nu\lambda\rho}(x_0) \chi_0^\mu \chi_0^\nu \chi_0^\lambda \chi_0^\rho \right] \prod_{k \neq 0} \frac{2\pi/\beta}{(2\pi/\beta)^2} \\
= \int \prod_{\mu=1}^{n} \left[ \frac{dx_0^\mu}{(2\pi)^{1/2}} d\chi_0^\mu d\chi_0^\mu \right] \exp \left[ -\frac{1}{4} \beta R_{\mu\nu\lambda\rho}(x_0) \chi_0^\mu \chi_0^\nu \chi_0^\lambda \chi_0^\rho \right]. \tag{4.25}
\]

Further, rescaling the constant fermionic modes by a factor of \( \beta^{-1/4} \) and absorbing \( \beta \),

\[
\chi_0^\mu \rightarrow \beta^{-1/4} \chi_0^\mu, \quad \chi_0^\mu \rightarrow \beta^{-1/4} \chi_0^\mu,
\]

we obtain

\[
\text{Tr} \left[ (-1)^F \exp(-\beta H) \right] = \frac{1}{(2\pi)^{n/2}} \int \prod_{\mu=1}^{n} dx_0^\mu \int \prod_{\nu=1}^{n} d\chi_0^\nu d\chi_0^\nu \exp \left[ -\frac{1}{4} R_{\mu\nu\lambda\rho}(x_0) \chi_0^\mu \chi_0^\nu \chi_0^\lambda \chi_0^\rho \right]. \tag{4.27}
\]

In the case that \( M_n \) is an even dimensional manifold, i.e., \( n = 2k \), expand the exponential function of the Grassmann variables such that,

\[
\exp \left[ -\frac{1}{4} R_{\mu\nu\lambda\rho}(x_0) \chi_0^\mu \chi_0^\nu \chi_0^\lambda \chi_0^\rho \right] = 1 - \frac{1}{4} R_{\mu\nu\lambda\rho}(x_0) \chi_0^\mu \chi_0^\nu \chi_0^\lambda \chi_0^\rho + \cdots + \frac{1}{(n/2)!} \left[ R_{\mu\nu\lambda\rho}(x_0) \chi_0^\mu \chi_0^\nu \chi_0^\lambda \chi_0^\rho \right]^{n/2}. \tag{4.28}
\]

According to the integration property of the Grassmann variables, only the last term gives a non-vanishing contribution, thus we have

\[
\text{Tr} \left[ (-1)^F \exp(-\beta H) \right] = \frac{1}{(2\pi)^{n/2}} \int d(\text{vol}) \int \left\{ \prod_{\mu=1}^{n} d\chi_0^1 d\chi_0^2 d\chi_0^2 \cdots d\chi_0^n d\chi_0^n \frac{1}{(n/2)!} \right\} \\
\times \left[ R_{\mu\nu\lambda\rho}(x_0) \chi_0^\mu \chi_0^\nu \chi_0^\lambda \chi_0^\rho \right]^{n/2} \\
= \frac{(-1)^{n/2}}{2^{n(n/2)!\pi^{n/2}}} \int_M d(\text{vol}) \epsilon_{\mu_1 \nu_1 \cdots \mu_n \nu_n} \epsilon_{\lambda_1 \rho_1 \cdots \lambda_n \rho_n} R_{\mu_1 \nu_1 \lambda_1 \rho_1} \cdots R_{\mu_n \nu_n \lambda_n \rho_n}, \tag{4.29}
\]

where \( d(\text{vol}) = dx_0^1 \cdots dx_0^n \) is the volume element of \( M \).
In the case that $M$ is a odd-dimensional manifold, i.e., $n = 2k + 1$, it is impossible to saturate the Grassmann integrals with any term appearing in the expansion of the exponential functions of the Grassman variables. Therefore, by the integration property of the Grassmann variables introduced in Appendix C, we obtain

$$\text{Tr} \left[ (-1)^F \exp(-\beta H) \right] = 0 \quad \text{when} \quad n = 2k + 1. \quad (4.30)$$

Eqs. (4.29) and (4.30) are the Gauss-Bonnet-Chern-Avez formula.

### 4.3 Hirzebruch Signature Theorem from 0+1-dimensional N=1 Supersymmetric Sigma Model

#### 4.3.1 Identify Signature Complex from Quantum N=1 Supersymmetric Sigma Model

Use the same model described by the Lagrangian (4.1) or (4.6) to identify the signature complex. It is easy to see that when $n$ is even, the Lagrangian (4.6) has a discrete symmetry: it is invariant under the following exchange of the anti-commutative variables,

$$\chi^\mu \rightarrow \chi^{*\mu}, \quad \chi^{*\mu} \rightarrow -\chi^\mu, \quad \mu = 0, 1, \cdots, n - 1. \quad (4.31)$$

At the quantum level, both $\chi^\mu$ and $\chi^{*\mu}$ become operators. This implies that there must exist a unitary operator in the Hilbert space to implement the discrete symmetry transformation. Denoting the operator as $U_5$, we have [15, 16, 17]

$$U_5 \chi^\mu U_5^{-1} = \chi^{*\mu}, \quad U_5 \chi^{*\mu} U_5^{-1} = -\chi^\mu, \quad (4.32)$$

and the Hamiltonian is invariant under the transformation,

$$U_5 H U_5^{-1} = H, \quad (4.33)$$

which can be verified explicitly.

From Eqs. (4.11) and (4.32), we obtain

$$U_5 Q U_5^{-1} = Q^*, \quad \text{and} \quad U_5 Q^* U_5^{-1} = -Q, \quad (4.34)$$

that is,

$$U_5 Q = Q^* U_5, \quad \text{and} \quad U_5 Q^* = -Q U_5. \quad (4.35)$$

Eq. (4.31) or (4.32) shows that the four successive actions of $U_5$ should be an identity operator,

$$U_5^4 = I. \quad (4.36)$$

In the following, observe the operation of $U_5$ on a state $|\Psi_p\rangle$ in the Hilbert space to find the geometric correspondence. First, from Eq. (4.32), $U_5$ converts a creation operator $\chi^{*\mu}$ of fermions into a destruction operator $\chi^\mu$, and vice-versa, while $|\Omega\rangle$ is a vacuum state with no fermion. Therefore, the action of $U_5$ should lead to a fully-filled fermionic state, i.e.,

$$U_5 |\Omega\rangle = |\Psi_n\rangle = \chi^{*1} \chi^{*2} \cdots \chi^{*n} |\Omega\rangle. \quad (4.37)$$
Then the action of \( \mathcal{U}_5 \) on \( |\Psi_p\rangle \) is

\[
\mathcal{U}_5 |\Psi_p\rangle = U\omega_{\mu_1\mu_2...\mu_p}(x)\chi^{*\mu_1}\chi^{*\mu_2}...\chi^{*\mu_p}|\Omega\rangle = U\omega_{\mu_1\mu_2...\mu_p}(x)\mathcal{U}_5|\Psi_p\rangle = U\mathcal{U}_5\mathcal{U}_5^{-1}U\mathcal{U}_5^{-1}U|\Psi_p\rangle.
\]

This means that \( \mathcal{U}_5 \) converts \( |\Psi_p\rangle \) into \( |\Psi_{n-p}\rangle \). Hence, it can be identified with the Hodge star operator in the differential manifold \( M_n \),

\[
\mathcal{U}_5 \leftrightarrow \ast.
\]

Second, when \( \mathcal{U}_5 \) acts on a \( p \)-fermion state \( |\Psi_p\rangle \), define an operator \( \tilde{\mathcal{U}} \) such that

\[
\tilde{\mathcal{U}}_5 = i^{n/2+p(p-1)}U\mathcal{U}_5.
\]

Note that there exists

\[
\tilde{\mathcal{U}}_5 = \mathcal{U}_5 \quad \text{when} \quad n = 4k, \ p = n/2 = 2k.
\]

Then we find that when \( n = 4k \ (k \in \mathbb{Z}^+) \),

\[
\tilde{\mathcal{U}}_5^2 = I.
\]

Thus the Hilbert space when \( n = 4k \) can be decomposed into two subspaces with respect to the eigenvalues of \( \tilde{\mathcal{U}}_5 \):

\[
\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,
\]

\[
\mathcal{H}_+ = \{|\Psi_p\rangle | \tilde{\mathcal{U}}_5|\Psi_p\rangle = |\Psi_p\rangle, \ 0 \leq p \leq n \},
\]

\[
\mathcal{H}_- = \{|\Psi_p\rangle | \tilde{\mathcal{U}}_5|\Psi_p\rangle = -|\Psi_p\rangle, \ 0 \leq p \leq n \}.
\]

Third, using Eq. (4.35), we can show

\[
\tilde{\mathcal{U}}_5Q = -Q^*\tilde{\mathcal{U}}_5, \quad \text{and} \quad \tilde{\mathcal{U}}_5Q^* = -Q\tilde{\mathcal{U}}_5.
\]

Hence

\[
\tilde{\mathcal{U}}_5(Q + Q^*) = -(Q + Q^*)\tilde{\mathcal{U}}_5,
\]

\[
\tilde{\mathcal{U}}_5(Q - Q^*) = (Q - Q^*)\tilde{\mathcal{U}}_5.
\]

Thus \( Q + Q^* \) maps \( \mathcal{H}_+ \) into \( \mathcal{H}_- \) and vice-versa,

\[
Q + Q^* : \mathcal{H}_\pm \rightarrow \mathcal{H}_{\mp}.
\]

Finally, let \( |\Psi(E,+1)\rangle \) and \( |\Psi(E,-1)\rangle \) denote the eigenvectors with eigenvalues \( E \) of the Hamiltonian operator \( H \) and the eigenvalues \( \pm 1 \) of \( \mathcal{U}_5 \),

\[
H|\Psi(E, \pm 1)\rangle = E|\Psi(E, \pm 1)\rangle, \quad \text{and} \quad \mathcal{U}_5|\Psi(E, \pm 1)\rangle = \pm|\Psi(E, \pm 1)\rangle.
\]
Then from (4.45), if $E \neq 0$, $(Q + Q^*)|\Psi(E, \pm 1)\rangle$ is a state vector with the same $E \neq 0$ but opposite eigenvalue of $\tilde{U}_5$, since

$$H (Q + Q^*) |\Psi(E, \pm 1)\rangle = (Q + Q^*) H |\Psi(E, \pm 1)\rangle = E (Q + Q^*) |\Psi(E, \pm 1)\rangle,$$

$$\tilde{U}_5 (Q + Q^*) |\Psi(E, \pm 1)\rangle = -(Q + Q^*) \tilde{U}_5 |\Psi(E, \pm 1)\rangle = \mp (Q + Q^*) |\Psi(E, \pm 1)\rangle. \quad (4.48)$$

Thus all state vectors with $E \neq 0$ appear in pairs of opposite $\tilde{U}_5$ eigenvalues, and only the state vectors with $E = 0$ need not to appear in $\tilde{U}_5$ pairs. Therefore, we can identify

$$(\mathcal{H}_\pm, Q + Q^*) \quad (4.49)$$

as the signature complex in differential form theory when $n = 4k$, and $p = n/2 = 2k$. The index for the complex is

$$\text{Tr} (\mathcal{U}_5) = n^{E=0} (\mathcal{U}_5 = +1) - n^{E=0} (\mathcal{U}_5 = -1) = \tau (M), \quad (4.50)$$

where $n^{E=0} (\mathcal{U}_5 = \pm 1)$ denotes the number of $p = 2k$-fermion states in $\mathcal{H}_\pm$ with $E = 0$.

Finally, it should be emphasized that like $\text{Tr} (-1)^F$, the trace $\text{Tr} (\mathcal{U}_5)$ is ill-defined due to the infinite dimensionality of the Hilbert space. Therefore, when calculating $\text{Tr} (\mathcal{U}_5)$ we use the regularized definition,

$$\text{Tr} (\mathcal{U}_5) |_R = \text{Tr} (\mathcal{U}_5 e^{-\beta H}), \quad \beta > 0. \quad (4.51)$$

The result should be independent of $\beta$ since its only contribution is from the states with $E = 0$.

### 4.3.2 Derivation of Hirzebruch Signature Theorem

In the following we calculate $\text{Tr} (\mathcal{U}_5 e^{-\beta H})$ to derive the Hirzebruch signature theorem. Adopting the Lagrangian (4.1) rather than the Lagrangian (4.6), we have from Eq. (4.5),

$$\psi^\mu_1 = \frac{1}{\sqrt{2}} (\chi^\mu + \chi^*\mu), \quad \text{and} \quad \psi^\mu_2 = \frac{i}{\sqrt{2}} (\chi^\mu - \chi^*\mu). \quad (4.52)$$

Hence the discrete symmetry given in Eq. (4.31) leads to the following symmetry transformations on $\psi^\mu_1$ and $\psi^\mu_2$,

$$\psi^\mu_1 \rightarrow -\psi^\mu_1, \quad \psi^\mu_2 \rightarrow \psi^\mu_2. \quad (4.53)$$

To be compatible with this symmetry, when we use the functional integral to evaluate $\text{Tr} (\mathcal{U}_5 e^{-\beta H})$, $\psi^\mu_1(t)$ must be integrated over with anti-periodic boundary condition (A.B.C.), and $\psi^\mu_2(t)$ integrated over periodic boundary conditions (P.B.C.) in $\beta$. The integration over the commutative variables $x^\mu(t)$ is always taken with periodic boundary conditions. Therefore, we have

$$\text{Tr} (\mathcal{U}_5 e^{-\beta H}) = \int_{\text{P.B.C.}} [dx^\mu d\psi^\mu_2] \int_{\text{A.B.C.}} [d\psi^\mu_1] \exp \left(- \int_0^\beta dt L [x, \psi_1, \psi_2] \right). \quad (4.54)$$

The P.B.C for $\psi^\mu_2(t)$ and $x^\mu(t)$ and the A.B.C. for $\psi^\mu_1(t)$, i.e.

$$x^\mu(0) = x^\mu(\beta), \quad \psi^\mu_2(0) = \psi^\mu_2(\beta), \quad \text{and} \quad \psi^\mu_1(0) = -\psi^\mu_1(\beta). \quad (4.55)$$
mean that there exist the following Fourier series expansions,

\[ x^\mu(t) = \sum_{k=-\infty}^{\infty} x_k^\mu \exp \left( \frac{2k\pi \imath}{\beta} t \right) = x_0^\mu + \sum_{k \neq 0} x_k^\mu \exp \left( \frac{2k\pi \imath}{\beta} t \right) \]

\[ \psi_2^\mu(t) = \sum_{k=-\infty}^{\infty} \psi_{2k}^\mu \exp \left( \frac{2k\pi \imath}{\beta} t \right) = \psi_{2,0}^\mu + \sum_{k \neq 0} \psi_{2,k}^\mu \exp \left( \frac{2k\pi \imath}{\beta} t \right) \]

\[ \psi_1^\mu(t) = \sum_{k \neq 0} \psi_{1,k}^\mu \exp \left( \frac{(2k+1)\pi \imath}{\beta} t \right) = 0 + \psi_1^\mu(t), \quad (4.56) \]

where \( k \in \mathbb{Z} \). Note that because all variables are real,

\[ x^\mu(t) = x^{* \mu}(t), \quad \psi_2^\mu(t) = \psi_2^{* \mu}(t), \quad \text{and} \quad \psi_1^\mu(t) = \psi_1^{* \mu}(t), \quad (4.57) \]

and there should exist

\[ x_k^{* \mu} = x_{-k}^\mu, \quad \psi_{1,k}^{* \mu} = \psi_{1,-k}^\mu, \quad \text{and} \quad \psi_{2,k}^{* \mu} = \psi_{2,-k}^\mu. \quad (4.58) \]

In the following we evaluate the functional integral (4.54) around the constant configuration

\[ (x^\mu, \psi_1^0, \psi_2^0) = (x_0^\mu, 0, \psi_{2,0}^\mu). \quad (4.59) \]

It is easy to see from the Lagrangian (4.1) that on the constant configuration (4.59),

\[ L[x_0^\mu, 0, \psi_{2,0}^\mu] = 0. \quad (4.60) \]

Substituting the expansions (4.56) around the background of the constant configuration (4.59) into the Lagrangian (4.1), we expand the Lagrangian to the second order of \( u^\mu(t), \psi_1^\mu(t), \) and \( \eta^\mu(t) \), [17]

\[ L = L^{(2)} + \mathcal{O}(\beta), \]

\[ L^{(2)} = \frac{1}{2} g_{\mu\nu}(x_0) \frac{du^\mu}{dt} \frac{du^\nu}{dt} + \frac{i}{4} R_{\mu\nu\lambda\rho} \psi_0^\mu \psi_0^\nu u^\lambda \frac{du^\rho}{dt} + \frac{i}{2} g_{\mu\nu}(x_0) \psi_1^\mu \frac{d\eta^\nu}{dt} + \frac{i}{4} R_{\mu\nu\lambda\rho} \psi_0^\mu \psi_2^\nu \psi_1^\lambda \psi_1^\rho + \frac{i}{2} g_{\mu\nu}(x_0) \eta^\mu \frac{d\eta^\nu}{dt}. \quad (4.61) \]

In deriving \( L^{(2)} \), we have discarded some total derivative terms due to the periodic boundary conditions, and made use of the classical equations of motion for \( x_\mu, \psi_1^\mu, \) and \( \psi_2^\mu \). Note that the constant configuration (4.59) trivially satisfies the classical equations of motion, and consequently, the terms which are linear in \( u^\mu, \eta^\mu \) and \( \psi_1^\mu \) vanish.

We define

\[ X^a = e^a_\mu(x_0) X^\mu, \quad \text{and} \quad X^\mu = \left( u^\mu, \eta^\mu, \psi_1^\mu, \psi_{2,0}^\mu \right), \quad (4.62) \]

where \( e^a_\mu(x) \) are vierbein shown in Eq. (2.17). The Lagrangian (4.61) becomes

\[ L^{(2)} = \frac{1}{2} \delta_{ab} \frac{du^a}{dt} \frac{du^b}{dt} + \frac{i}{4} R_{abcd} \psi_{2,0}^c \psi_{2,0}^d u^a \frac{du^b}{dt} + \frac{i}{2} \delta_{ab} \eta^a \frac{d\eta^b}{dt} + \frac{i}{2} \delta_{ab} \psi_{1}^a \frac{d\psi_{1}^b}{dt} \psi_{1}^b + \frac{1}{4} R_{abcd} \psi_{2,0}^c \psi_{2,0}^d \psi_{1}^a \psi_{1}^b. \quad (4.63) \]
Derivation of Index Theorem

\[ S_E^{(2)} = \int_0^\beta dt L^{(2)} = \int_0^\beta dt \left\{ \frac{1}{2} u^a \left[ -\frac{d}{dt^2} + \frac{i}{2} R_{abcd}(x_0) \psi_{2,0}^c \psi_{2,0}^d \right] u^b \right\} + \frac{1}{2} \eta^a \left[ i \delta_{ab} \frac{d}{dt} \right] \eta^b + \frac{1}{2} u^a \left[ i \delta_{ab} \frac{d}{dt} + \frac{1}{2} R_{abcd}(x_0) \psi_{2,0}^c \psi_{2,0}^d \right] \psi^b \right\}. \] (4.64)

Therefore by the formulas (3.82) and (B.29), the index of the signature complex from Eq. (4.54) is

\[ \text{Tr} \left( U_5 e^{-\beta H} \right) = \int_n \prod_{\mu=1}^n \frac{dx_0^\mu}{(2\pi)^{1/2}} \prod_{\nu=1}^n d\psi_{2,0}^{\nu} \left[ du \, d\eta \, d\psi_1 \right] \exp \left\{ -S_E^{(2)} \right\} = \int \prod_{\mu=1}^n \frac{dx_0^\mu}{(2\pi)^{1/2}} \prod_{\nu=1}^n d\psi_{2,0}^{\nu} \left[ \frac{\det' \left( i\delta_{ab} \frac{d}{dt} \right) \det (i\delta_{ab} \frac{d}{dt} + M_{ab})}{\det' \left( -\delta_{ab} \frac{d^2}{dt^2} + iM_{ab} \frac{d}{dt} \right) \psi^b \psi^c} \right]^{1/2}, \] (4.65)

where

\[ M_{ab} \equiv \frac{1}{2} R_{abcd}(x_0) \psi_{2,0}^c \psi_{2,0}^d. \] (4.66)

In (4.65), \( \det' \) means that the zero modes have been taken away and that the determinant is evaluated over the configurations satisfying the P.B.C, whereas the determinant, \( \det (i\delta_{ab} \frac{d}{dt} + M_{ab}) \), is calculated with the A.B.C.

Because \( M_{ab} \) form an \( n \times n \) antisymmetric matrix, \( M = -M^t \). According to linear algebra [10], \( M_{ab} \) can be converted into a standard skew diagonal form by a similar transformation,

\[ (M_{ab}) = QSQ^{-1} \] (4.67)

where

\[ S = \begin{bmatrix} 0 & x_1 & \cdots & \cdots & 0 \\ -x_1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_{n/2} & \cdots & \cdots & 0 \\ -x_{n/2} & 0 & \cdots & \cdots & 0 \end{bmatrix}. \] (4.68)

Then we have the following:

\[ \left[ \det' \left( i\delta_{ab} \frac{d}{dt} \right) \right]^{1/2} = \int [d\eta^a] \exp \left\{ -\frac{1}{2} \int_0^\beta dt \eta^a \left( i\delta_{ab} \frac{d}{dt} \right) \eta^b \right\} = \int \prod_{k \neq 0} \prod_{a=1}^n d\psi_{2,k}^a \exp \left\{ -\sum_{k \neq 0} \pi k \psi_{2,k}^a \psi_{2,-k}^a \right\} = \int \prod_{k > 0} \prod_{a=1}^n \left[ d\psi_{2,k}^a d\psi_{2,-k}^a \right] \exp \left\{ -\sum_{k > 0} 2\pi k \psi_{2,k}^a \psi_{2,-k}^a \right\} = \prod_{a=1}^n \prod_{k > 0} \left( 2\pi k \right) = \prod_{k > 0} \left( 2\pi k \right)^n. \] (4.69)
\[
\left[ \det \left( i \delta_{ab} \frac{d}{dt} + M_{ab} \right) \right]^{1/2} = \left[ \det Q \left( i \delta_{ab} \frac{d}{dt} + S_{ab} \right) Q^{-1} \right]^{1/2}
\]
\[
= \left[ \det \left( i \delta_{ab} \frac{d}{dt} + S_{ab} \right) \right]^{1/2} = \int [d\psi_1] \exp \left[ -\frac{1}{2} \int_0^\beta dt \psi_1 \left( i \delta_{ab} \frac{d}{dt} + S_{ab} \right) \psi_1^b \right]
\]
\[
= \int \prod_{a,b=1}^n \prod_k d\psi_{1,k} \exp \left( -\sum_k \psi_{1,k} \psi_{1,-k} \left( (2k + 1) \pi \delta_{ab} + S_{ab} \right) \right)
\]
\[
= \int \prod_{p=1}^{n/2} \prod_k d\psi_{1,k} \exp \left\{ -\sum_k \sum_{p=1}^{n/2} \psi_{1,k} \psi_{1,-k} \left[ (2k + 1)^2 \pi^2 + x_p^2 \right] \right\}
\]
\[
= \prod_{p=1}^{n/2} \prod_{k \geq 0} d\psi_{1,k} d\psi_{1,-k} \exp \left\{ -2 \sum_k \psi_{1,k} \psi_{1,-k} \left[ (2k + 1)^2 \pi^2 + x_p^2 \right] \right\}
\]
\[
= \prod_{p=1}^{n/2} \prod_{k \geq 0} \left[ (2k + 1)^2 \pi^2 + x_p^2 \right]
\]
\[
= \prod_{k \geq 0} 2^{n/2} \left[ (2k + 1)^2 \pi^2 \right]^{n/2} \prod_{p=1}^{n/2} \prod_{k \geq 0} \left[ 1 + \frac{(x_p/2)^2}{\pi^2(k + 1/2)^2} \right]. \quad (4.70)
\]

\[
\left[ \det' \left( i \delta_{ab} \frac{d}{dt} + M_{ab} \right) \right]^{-1/2} = \left[ \det' \left( -\delta_{ab} \frac{d^2}{dt^2} + iS_{ab} \frac{d}{dt} \right) \right]^{1/2}
\]
\[
= \int [du] \exp \left[ -\frac{1}{2} \int_0^\beta dt u^a \left( -\delta_{ab} \frac{d^2}{dt^2} + iS_{ab} \frac{d}{dt} \right) u^b \right]
\]
\[
= \int \prod_{a,b=1 \neq 0}^n \prod_{k \geq 0} \frac{dx_{k}^a}{(2\pi)^{1/2}} \exp \left( -\sum_{k \neq 0} x_{k}^a x_{-k}^b \left[ (2\pi k)^2 \delta_{ab} + 2\pi k S_{ab} \right] \right)
\]
\[
= \prod_{p=1}^{n/2} \prod_{k > 0} \frac{dx_{k}^p}{2\pi} \exp \left\{ -2 \sum_{k > 0} x_{k}^p x_{-k}^p \left[ (2\pi k)^4 + (2\pi k)^2 x_p^2 \right] \right\}
\]
\[
= \prod_{p=1}^{n/2} \prod_{k > 0} \frac{dx_{k}^p dx_{-k}^p}{2\pi} \exp \left\{ -2 x_{k}^p x_{-k}^p (2\pi k)^2 \left[ (2\pi k)^2 + x_p^2 \right] \right\}
\]
\[
= \prod_{p=1}^{n/2} \prod_{k > 0} \frac{1}{2(2\pi k)^2 \left[ (2\pi k)^2 + x_p^2 \right]} = \prod_{k > 0} \frac{1}{2^{n/2}(2\pi k)^{2n}} \prod_{p=1}^{n/2} \prod_{k > 0} \left[ 1 + \frac{(x_p/2)^2}{\pi^2 k^2} \right]. \quad (4.71)
\]

The multiple variable integral formulas for commutative variables and the Grassmann ones have been used in calculating these determinants.

Substituting the results (4.69), (4.70), and (4.71) into (4.65) and discarding some irrelevant
factors that can be absorbed into the normalization factor of the functional integral, we obtain

$$\text{Tr} \left( U e^{-\beta H} \right) = \int \prod_{\mu=1}^{n} \frac{dx_0^\mu}{(2\pi)^{1/2}} \prod_{\mu=1}^{n} d\psi_{2,0}^\mu \prod_{p=1}^{\frac{n}{2}} \prod_{k>0} \left( 1 + \frac{(x_p/2)^2}{\pi^2(k+1/2)^2} \right) \prod_{p=1}^{\frac{n}{2}} \prod_{k>0} \left( 1 + \frac{(x_p/2)^2}{\pi^2k^2} \right)$$

$$= \frac{1}{(2\pi)^{n/2}} \int \prod_{\mu=1}^{n} dx_0^\mu \int \prod_{\mu=1}^{n} d\psi_{2,0}^\mu \prod_{p=1}^{\frac{n}{2}} \cosh \left( \frac{x_p}{2} \right) \prod_{p=1}^{\frac{n}{2}} \left[ \sinh \left( \frac{x_p}{2} \right) / \left( \frac{x_p}{2} \right) \right]$$

$$= \frac{1}{(2\pi)^{n/2}} \int d(\text{vol}) \prod_{p=1}^{\frac{n}{2}} \frac{x_p/2}{\tanh \left( \frac{x_p}{2} \right)} = \int L(M_a) = \tau(M_a). \quad (4.72)$$

Eq. (4.72) used the product representations of cosh $x$ and sinh $x$ [28], the identification

$$M_{ab} = \frac{1}{2} R_{abcd}(x_0) \psi_0^c \not{\psi}_0^d \leftrightarrow \Omega_{ab} = \frac{1}{2} R_{abcd} e^c \wedge e^d, \quad (4.73)$$

and Eqs. (4.67) and (4.68),

$$(M_{ab}) = \bigoplus_{p=1}^{\frac{n}{2}} (x_p \epsilon_{p,q}) \quad a, b = 1, \ldots, n, \quad p, q = 1, \ldots, \frac{n}{2}. \quad (4.76)$$

Eq. (4.72) is the Hirzebruch signature theorem reproduced from the supersymmetric quantum mechanical model (4.1).

### 4.4 Index Theorem for A-roof Genus from 0+1-dimensional N=1/2 Supersymmetric Sigma Model

#### 4.4.1 Designing Spin Complex from Quantum N=1/2 Supersymmetric Sigma Model

To get the spin complex from supersymmetric quantum mechanics, we need to slightly modify the model (4.1). Impose the following condition on the Lagrangian (4.1),

$$\psi_1^\mu = \psi_2^\mu = \frac{1}{\sqrt{2}} \chi^\mu. \quad (4.74)$$

Then, due to the symmetric property of the Riemannian curvature listed in (2.32), the term containing the curvature vanishes and the Lagrangian becomes

$$L = \frac{1}{2} g_{\mu\nu}(x) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + \frac{i}{2} g_{\mu\nu}(x) \chi^\mu \frac{D\chi^\nu}{dt}. \quad (4.75)$$

Correspondingly, in the supersymmetry transformations (4.3), one should take $\epsilon_1 = -\epsilon_2 = -1/\sqrt{2} \eta$. Then the supersymmetry transformations reduce (4.3) to

$$\delta x^\mu = i \epsilon \chi^\mu, \quad \text{and} \quad \delta \psi^\mu = -\dot{\psi}^\mu \epsilon. \quad (4.76)$$
By the Nöther theorem, we can obtain one (real) supersymmetry generator $Q$ from the Lagrangian (4.75),
\[ Q = \chi^\mu g_{\mu\nu} \dot{x}^\nu. \tag{4.77} \]
This type of supersymmetry is usually called $N = 1/2$ supersymmetry since $2N = 1$. The Hamiltonian is
\[ H = Q^2. \tag{4.78} \]

As in Sects. 4.2 and 4.3, use the standard canonical quantization procedure to construct quantum theory. The coordinates are $(x^\mu, \chi^\mu)$ and the corresponding canonical conjugate momenta are
\[ p^\mu = \frac{\partial L}{\partial \dot{x}_\mu} = g^{\mu\nu} \left( \dot{x}_\nu + \frac{i}{2} \Gamma^\lambda_{\nu\rho} g_{\lambda\sigma} \chi^\sigma \chi^\rho \right), \quad \text{and} \]
\[ \pi^\mu = \frac{\partial L}{\partial \dot{\chi}_\mu} = \frac{i}{2} g^{\mu\nu} \dot{\chi}_\nu. \tag{4.79} \]
Then we have the canonical commutation relation,
\[ [x_\mu, p_\nu] = ig^{\mu\nu} (x), \tag{4.80} \]
and anticommutation relation,
\[ \{ \chi^\mu, \chi^\nu \} = g^{\mu\nu} (x). \tag{4.81} \]

For the commutative variables $x^\mu$ and $p^\mu$, choose the coordinate representation, then $p_\mu = -i \frac{\partial}{\partial x^\mu}$. To find the representations for the anti-commutative variables $\chi^\mu$, we need to express them in terms of the orthonormal vierbein defined at each point of the Riemannian manifold. The reason for this is that the isometry group for the Riemannian manifold is $GL(n, \mathbb{R})$, which has no spinor representation. If we express various vectors such as the metric, connection and various vectors on $M$ in the vierbein formalism, the isometry group is the local Lorentz group $SO(n)$, which admits a spinor representation. Therefore, we define
\[ \chi^a \equiv e^a_\mu (x) \chi^\mu. \tag{4.82} \]
Then the anticommutation relation (4.81) becomes
\[ \{ \chi^a, \chi^b \} = \delta^{ab}, \tag{4.83} \]
which is precisely the fundamental relation of the Clifford algebra. Thus we can choose the representation
\[ \chi^a = \frac{\gamma^a}{\sqrt{2}}, \tag{4.84} \]
with $\gamma^a$ being $2^{n/2} \times 2^{n/2}$ matrices, as shown in Appendix C. Consequently, the conjugate momenta operator $p_\mu$ becomes
\[
p_\mu = -i \frac{\partial}{\partial x^\mu} = g_{\mu\nu} (x) \dot{x}^\nu + \frac{i}{4} \omega_{\mu\ab}[\chi^a, \chi^b]
= g_{\mu\nu} (x) \dot{x}^\nu - \frac{1}{8} \omega_{\mu\ab}[\chi^a, \chi^b], \tag{4.85} \]
where we have used the relation between the Christoffel symbol $\Gamma^\nu_{\mu\rho}$ and the spin correction $\omega_{\mu ab}$ given in Eq. (2.37). Using (4.85) we can express the supersymmetry generator (4.77) as,

$$Q = -\frac{i}{\sqrt{2}}\gamma^\mu \left( \frac{\partial}{\partial x^\mu} + \frac{1}{8} \omega_{ab\mu}[\gamma^a, \gamma^b] \right) = -\frac{i}{\sqrt{2}} \gamma^\mu D_\mu \equiv -\frac{i}{\sqrt{2}} D,$$

(4.86)

where $\gamma^\mu \equiv \gamma^a E^\mu_a(x)$. Eq. (4.86) shows that the supersymmetry generator is identified with the Dirac operator. Then the representation of the Hamiltonian operator is $H = \frac{1}{2} D^2$.

The following is a construction of the Hilbert space based on the above representations of the operator. Choose the coordinate representation for $x^\mu$ and $p_\mu$, and choose the $\gamma$-matrix representations for $\psi^\mu$ since they satisfy the Clifford algebra whose representation space is the space of spinor functions. Therefore, the Hilbert space can be identical to the spinor space formed by the spinor functions $\Psi(x)$,

$$\mathcal{H} = \left\{ \Psi(x) | \Psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_{n/2}(x) \end{bmatrix} \right\}. \quad (4.87)$$

Therefore, we have obtained the following identification between the supersymmetric quantum mechanics (4.75) and the spin complex:

$$\begin{align*}
(-1)^F &\leftrightarrow \gamma_{n+1}, \\
\mathcal{H}_B &\leftrightarrow S^+, \\
\mathcal{H}_F &\leftrightarrow S^-, \\
Q &\leftrightarrow iD.
\end{align*} \quad (4.88)$$

Hence, the index for the spin complex can be obtained from supersymmetric quantum mechanics described by the Lagrangian (4.75), and it is is

$$\text{Index of } iD = \text{Tr} \left[ (-1)^F e^{-\beta H} \right], \quad (4.89)$$

where, as above, $\beta$ is an arbitrary positive parameter and $H$ is the Hamiltonian operator.

### 4.4.2 Derivation of Index Theorem for A-roof Genus

The procedure of computing $\text{Tr} \left[ (-1)^F e^{-\beta H} \right]$ to get the index of the Dirac operator is identical to the procedure of calculating $\text{Tr} \left( \mathcal{U}_0 \exp [ -\beta H] \right)$. That is, start from the Lagrangian (4.75) and expand it to the second order of variables around the trivial constant solution to the classical equations of motion. Then, calculate the functional integration over $(x^\mu, \chi^\mu)$. Note that the functional integral should be evaluated with both the variables satisfying periodic boundary conditions. Therefore, the index of the spin complex is

$$\begin{align*}
\text{Ind}(i\mathcal{D}) &= \text{Tr} \left[ (-1)^F \exp (-\beta H) \right] = \int_{\text{P.B.C.}} [dx] [d\chi] \exp \left[ -\int_0^\beta dt L \right] \\
&= \frac{1}{(2\pi)^{n/2}} \int \prod_{\mu=1}^n dx_0^\mu \prod_{\mu=1}^n d\psi_0^\mu \left[ \frac{\det' \left( i\delta^{ab} \frac{d}{dt} \right)}{\det' \left( -\delta^{ab} d^2/dt^2 + iM_{ab}/dt \right)} \right]^{1/2}. \quad (4.90)
\end{align*}$$
As shown in Eq. (4.70), \( \text{det}' (i \delta^{ab} d/dt) \) is a constant factor, which can be canceled by the normalization factor of the functional integral. Thus, only the determinant in the denominator contributes to the index. Using the result displayed in Eq. (4.71), we obtain

\[
\text{Ind}(i D) = \frac{1}{(2\pi)^{n/2}} \int \prod_{\alpha=1}^{n} dx_{\alpha}^{\mu} \int \prod_{\mu=1}^{n} d\chi_{\alpha}^{\mu} \prod_{\rho=1}^{n/2} \frac{x_{\rho} / 2}{\sinh (x_{\rho} / 2)}
\]

\[
= \frac{1}{(2\pi)^{n/2}} \int d(\text{vol}) \prod_{\rho=1}^{n/2} \frac{x_{\rho} / 2}{\sinh (x_{\rho} / 2)} = \tilde{A}(M_{n}). \tag{4.91}
\]

Eq. (4.91) is the index theorem for the spin complex.

### 4.5 Riemann-Roch Theorem from 0+1-dimensional N=2 Supersymmetric Sigma Model

#### 4.5.1 Making Dolbeault Complex from Quantum N=2 Supersymmetric Sigma Model

To make the Dolbeault complex from a supersymmetric quantum mechanical system, we need to modify the model (4.1). The first step is to consider the \( n \)-dimensional manifold \( M \) as a Kähler manifold (\( n = 2k \)). Second, write the real coordinates \( x^{\mu} \), and the real Grassmann variables \( \psi^{a} \), into two complex counterparts according to the standard form of the complex structure shown in Sect. 2.1,

\[
(x^{\mu}) = (x^{1}, \ldots, x^{k}, x^{1+k}, \ldots, x^{2k}) \rightarrow (z^{\alpha}; \bar{z}^{\alpha}) = (z^{1}, \ldots, z^{k}; \bar{z}^{1}, \ldots, \bar{z}^{k}),
\]

\[
(\psi^{a}) = (\psi_{1}^{a}, \ldots, \psi_{k}^{a}, \psi_{1+k}^{a}, \ldots, \psi_{2k}^{a}) \rightarrow (\psi^{a}; \bar{\psi}^{a}) = (\psi_{1}^{a}, \ldots, \psi_{k}^{a}; \bar{\psi}_{1}^{a}, \ldots, \bar{\psi}_{k}^{a}),
\]

\[
\mu = 1, 2, \ldots, n, \quad \alpha = 1, 2, \ldots, k = n/2, \quad r = 1, 2, \tag{4.92}
\]

where

\[
z^{\alpha} = \frac{1}{\sqrt{2}} (x^{\alpha} + ix^{\alpha+k}), \quad \bar{z}^{\alpha} = \frac{1}{\sqrt{2}} (x^{\alpha} - ix^{\alpha+k});
\]

\[
\psi^{a}_{r} = \frac{1}{\sqrt{2}} (\psi_{1}^{a} + i\psi_{2}^{a}), \quad \bar{\psi}^{a}_{r} = \frac{1}{\sqrt{2}} (\psi_{1}^{a} - i\psi_{2}^{a}). \tag{4.93}
\]

Further, define

\[
\chi^{a} = \frac{1}{\sqrt{2}} (\psi_{1}^{a} + i\psi_{2}^{a}), \quad \chi^{*a} = \frac{1}{\sqrt{2}} (\psi_{1}^{a} - i\psi_{2}^{a}),
\]

\[
\bar{\chi}^{a} = \frac{1}{\sqrt{2}} (\bar{\psi}_{1}^{a} + i\bar{\psi}_{2}^{a}), \quad \bar{\chi}^{*a} = \frac{1}{\sqrt{2}} (\bar{\psi}_{1}^{a} - i\bar{\psi}_{2}^{a}). \tag{4.94}
\]

Then the Lagrangian (4.1) is converted into the following form [26, 27],

\[
L = g_{\alpha\beta}(z, \bar{z}) \frac{dz^{\alpha}}{dt} \frac{d\bar{z}^{\beta}}{dt} + ig_{\alpha\beta}(z, \bar{z}) \chi^{*a} \frac{D}{dt} \chi^{a} + ig_{\alpha\beta}(z, \bar{z}) \chi^{*a} \frac{D}{dt} \bar{\chi}^{a} + \frac{1}{4} R_{\alpha\beta\gamma\delta} \chi^{a} \chi^{b} \chi^{c} \bar{\chi}^{d}. \tag{4.95}
\]
The real two-component anti-commutative spinor parameter in (4.4) should be promoted a complex counterpart and be combined in the same way as \( \psi^\mu \) shown in (4.93) and (4.94),

\[
\eta = \frac{1}{\sqrt{2}} (\epsilon_1 + i\epsilon_2), \quad \eta^* = \frac{1}{\sqrt{2}} (\epsilon_1 - i\epsilon_2),
\]

\[
\bar{\eta} = \frac{1}{\sqrt{2}} (\overline{\epsilon_1} + i\overline{\epsilon_2}), \quad \bar{\eta}^* = \frac{1}{\sqrt{2}} (\overline{\epsilon_1} - i\overline{\epsilon_2}).
\]

(4.96)

Consequently, the supersymmetry transformations for the model (4.95) are listed as follows,

\[
\begin{align*}
\delta z^\alpha &= i\bar{\eta}^* \chi^\alpha + i\eta \chi^{*\alpha}, \\
\delta \bar{z}^\alpha &= i\eta^* \bar{\chi}^\alpha + i\bar{\eta} \bar{\chi}^{*\alpha}, \\
\delta \chi^\alpha &= -\eta^* \bar{z}^\alpha - i\eta \chi^{*\beta} \Gamma^\beta_{\alpha\gamma} \chi^\gamma, \\
\delta \bar{\chi}^\alpha &= -\bar{\eta}^* \bar{z}^\alpha - i\bar{\eta} \bar{\chi}^{*\beta} \Gamma^\beta_{\alpha\gamma} \bar{\chi}^\gamma
\end{align*}
\]

(4.97)

where \( \Gamma^\alpha_{\beta\gamma} \) and \( \Gamma^\gamma_{\beta\gamma} \) are given in Eq. (2.89).

Choose the canonical coordinates \((z^\alpha, \bar{z}^\alpha, \chi^\alpha, \bar{\chi}^\alpha)\), the corresponding conjugate momenta are, respectively,

\[
\begin{align*}
p_{\alpha} &= \frac{\partial L}{\partial \dot{z}^\alpha} = g_{\alpha\beta} \left( \dot{z}^\beta - \Gamma^\beta_{\gamma\delta} \bar{\chi}^{*\gamma} \chi^{\delta} \right), \\
\bar{p}_{\alpha} &= \frac{\partial L}{\partial \dot{\bar{z}}^\alpha} = g_{\bar{\alpha}\beta} \left( \dot{\bar{z}}^\beta - \Gamma^\beta_{\gamma\delta} \chi^{*\gamma} \bar{\chi}^{\delta} \right), \\
p_{\alpha} &= \frac{\partial L}{\partial \dot{\chi}^\alpha} = ig_{\alpha\beta} (z, \bar{z}) \chi^{*\beta}, \quad \text{and} \quad \bar{p}_{\alpha} = \frac{\partial L}{\partial \dot{\bar{\chi}}^\alpha} = ig_{\bar{\alpha}\beta} (z, \bar{z}) \bar{\chi}^{*\beta}.
\end{align*}
\]

(4.98)

Using the Nöther theorem, we can find four generators of supersymmetry transformations corresponding to the four parameters \( \eta, \bar{\eta}, \eta^*, \) and \( \bar{\eta}^* \),

\[
\begin{align*}
Q &= ig_{\alpha\beta} \chi^{*\alpha} p^\beta, \quad & Q &= -ig_{\beta\alpha} \chi^{*\beta} p^\alpha, \\
Q^* &= ig_{\bar{\alpha}\beta} \chi^{*\bar{\alpha}} \bar{p}^\beta, \quad & Q^* &= -ig_{\bar{\beta}\alpha} \chi^{*\bar{\beta}} \bar{p}^\alpha.
\end{align*}
\]

(4.99)

The quantum theory can be constructed by canonical quantization. First define the following operator commutation relations,

\[
\begin{align*}
[z^\alpha, p_{\beta}] &= i\delta^\alpha_{\beta}, \quad & [\bar{z}^\alpha, \bar{p}_{\beta}] &= i\delta^\alpha_{\beta}, \\
[z^\alpha, z^\beta] &= [\bar{z}^\alpha, \bar{z}^\beta] = [\bar{z}^\alpha, \bar{z}^\beta] = 0, \\
[p_{\alpha}, p_{\beta}] &= [\bar{p}_{\alpha}, \bar{p}_{\beta}] = [\bar{p}_{\alpha}, \bar{p}_{\beta}] = 0, \\
[z^\alpha, \bar{p}_{\beta}] &= [\bar{z}^\alpha, p_{\beta}] = 0;
\end{align*}
\]

(4.100)

and anticommutation ones,

\[
\begin{align*}
\{\chi^\alpha, \chi^{*\beta}\} &= g_{\alpha\beta}, \quad \{\bar{\chi}^\alpha, \chi^{*\beta}\} = g_{\bar{\alpha}\beta}, \\
\{\chi^\alpha, \chi_{\beta}\} &= \{\chi^\alpha, \chi^{*\beta}\} = \{\chi^{*\alpha}, \chi_{\beta}\} = \{\chi^{*\alpha}, \chi^{*\beta}\} = 0, \\
\{\chi^{*\alpha}, \chi_{\beta}\} &= \{\bar{\chi}^\alpha, \chi_{\beta}\} = \{\bar{\chi}^{*\alpha}, \bar{\chi}_{\beta}\} = \{\bar{\chi}^{*\alpha}, \bar{\chi}^{*\beta}\} = 0.
\end{align*}
\]

(4.101)
For the commutative operators, choose the coordinate representation, and for the commutative ones, choose the particle number representation. Then we have

\[ p_\alpha = -i \frac{\partial}{\partial z^\alpha}, \quad \text{and} \quad \bar{p}_\alpha = -i \frac{\partial}{\partial \bar{z}^\alpha}, \]  

where

\[ \chi^\alpha = \text{destruction operator of holomorphic fermions}, \]

\[ \chi^{\ast \alpha} = \text{creation operator of holomorphic fermions}, \]

\[ \bar{\chi}^\alpha = \text{destruction operator of anti-holomorphic fermions}, \]

\[ \chi^{\ast \beta} = \text{creation operator of anti-holomorphic fermions}. \] (4.103)

The Hilbert space is composed of the state vectors which can be called \((p,q)\)-fermion states,

\[ \mathcal{H} = \left\{ |\Psi_{p,q}\rangle \middle| |\Psi_{p,q}\rangle = \omega_{\alpha_1\ldots\alpha_p\bar{\beta}_1\ldots\bar{\beta}_q}(z,\bar{z})\chi^{\alpha_1}\cdots\chi^{\alpha_p}\bar{\chi}^{\beta_1}\cdots\bar{\chi}^{\beta_q}|\Omega\rangle, 0 \leq p, q \leq k \right\} \] (4.104)

where \( |\Omega\rangle \) is the \((p,q)\)-fermion vacuum state vector, defined by

\[ \chi^{\alpha}|\Omega\rangle = \chi^{\alpha*}|\Omega\rangle = 0. \] (4.105)

As in Sect. 4.2, we can easily show

\[ Q^2 = Q^*Q = \bar{Q}^2 = 0, \quad H = 2 \left( \bar{Q}Q^* + Q\bar{Q} \right) = 2 \left( \overline{Q}Q^* + Q^*\overline{Q} \right), \] (4.106)

and further,

\[ Q|\Psi_{p,q}\rangle \sim |\Psi_{p+1,q}\rangle, \quad \overline{Q}^*|\Psi_{p,q}\rangle \sim |\Psi_{p-1,q}\rangle; \]

\[ \overline{Q}|\Psi_{p,q}\rangle \sim |\Psi_{p,q+1}\rangle, \quad Q^*|\Psi_{p,q}\rangle \sim |\Psi_{p,q-1}\rangle. \] (4.107)

So we have the identifications

\[ Q \leftrightarrow \partial, \quad Q^* \leftrightarrow \partial^*; \]

\[ \mathcal{H} = \{ |\Psi_{p,q}\rangle \} \leftrightarrow \Lambda(M) = \bigoplus_{p,q=0}^{k} \Lambda^{p,q}(M), \]

\[ H = \{ Q, Q^* \} = \{ \overline{Q}, Q^* \} \leftrightarrow \Delta = \{ d, \delta \} = \{ \partial, \partial^* \} = \{ \overline{\partial}, \overline{\partial}^* \}, \]

\[ (-1)^{F_p} \leftrightarrow (-1)^p, \quad (-1)^{F_q} \leftrightarrow (-1)^q, \] (4.108)

where \( F_p \) and \( F_q \) are the operators of \( p \)-fermion and \( q \)-fermion numbers.

Now take \( p = 0 \) and consider a subspace of the Hilbert space which is composed only of \( q \)-fermion states,

\[ \mathcal{H}_A = \{ |\Psi_{0,q}\rangle, 0 \leq q \leq k \}. \] (4.109)

Then \((\overline{Q}, \mathcal{H}_A)\) is identified with the Dolbeault complex and \((\overline{\partial}, \oplus \Lambda^{0,q})\). Therefore, the analytic index of the Dolbeault complex is

\[ \text{Index of } \overline{\partial} = \text{Tr} \left[ (-1)^{F_q} e^{-\beta H} \right] = \sum_{q=0}^{n/2} (-1)^q \beta^{0,q}, \] (4.110)

where \( \beta > 0 \) and \( H \) is the Hamiltonian operator.
4.5.2 Derivation of Riemann-Roch Theorem

We briefly state how the Riemann-Roch theorem for the Dolbeault complex can be derived since the procedure is identical to that for the spin complex and the signature complex. We only need to restrict the calculation of $\text{Tr} \left[ (-1)^F \exp(-\beta H) \right]$ to the anti-holomorphic part of the Hilbert space. The computation is exactly the same as for that of the Dirac operator, and the non-trivial contribution to the index of $\partial$ comes only from the integration over $z^\alpha$ and $\bar{z}^\alpha$. Finally we obtain

$$\text{Ind}(\partial) = \frac{1}{(2\pi)^{n/2}} \int \prod_{\alpha=1}^{n/2} \left[ dz_0^\alpha \, d\bar{z}_0^\alpha \right] \prod_{\alpha=1}^{n/2} \left[ dx_0^{\ast \alpha} \, dx_0^{\delta} \right] \int \prod_{\alpha=1}^{n/2} e^{\omega_{\alpha}/2} \frac{\omega_{\alpha}/2}{\sinh(\omega_{\alpha}/2)}$$

$$= \frac{1}{(2\pi)^{n/2}} \int_M d\text{vol} \prod_{\alpha=1}^{n/2} \frac{\omega_{\alpha}}{1 - \exp(-\omega_{\alpha})} = \int_M \text{td}_c(M), \quad (4.111)$$

where $i\omega_{\alpha}$ is the eigenvalue of the antisymmetric matrix

$$\Omega_{\alpha\beta} = \frac{i}{2\pi} R_{\alpha\beta\gamma\delta} \lambda_0^{\ast \gamma} \bar{\lambda}_0^\delta. \quad (4.112)$$

Eq. (4.111) is the Riemann-Roch theorem for the Dolbeault complex.
Chapter 5

Summary and A Mathematical Conjecture

So far the Atiyah-Singer index theorem for four classical elliptic complexes and the relevant preliminary knowledge has been introduced. It has been shown in great detail how the Hilbert spaces of 0 + 1-dimensional supersymmetric nonlinear sigma models can be identical to the sections of fibre bundles for elliptic complexes, and how the generators of supersymmetry can be identified with differential operators. Further, the index theorems have been derived using powerful physical approaches – path integration and using structures of supersymmetric quantum theory. As Witten realized in the early 1980s [6, 7, 22], some supersymmetric quantum theories have provided a natural physical framework for studying the Atiyah-Singer index theorem and relevant mathematical problems. First, the Hilbert space of a supersymmetric quantum theory presents a graded structure. Hence an isomorphism with the exterior algebra can be established. Second, the generators of supersymmetry in a quantum mechanical system are nilpotent and transform the “bosonic” sector of the Hilbert space into the “fermionic” sector and vice versa. Hence, they can be identified as exterior differential operators and the relevant variants in various complexes. According to supersymmetry algebra, the Hamiltonian of the system, which in general determines all the physical processes, is identified with the Laplacian operator in elliptic complexes. Third, supersymmetry can make infinity reduce to finiteness. It is well known in physics that supersymmetry enforces the quantum contributions to a physical process coming from the paired states in the graded sectors to cancel. Therefore, the Hilbert space can effectively become a finite dimensional space composed of only non-paired states, which are eigenstates corresponding to the zero-eigenvalue of the Hamiltonian operator. The creation of finite dimensional reduction from an infinite dimensional Hilbert space is the most useful property of supersymmetry for mathematics.

However, the derivation of the Atiyah-Singer theorem by supersymmetric quantum mechanics relies on some ideas of physics. Because of this, it is somehow not mathematically rigorous. As mentioned in Chapter 1, the proof of the Atiyah-Singer index theorem using the approach of heat equations [4, 5] is based on positive-self adjoint operator theory and has been accepted as a mathematically rigorous proof. In this approach, the derivation of the index theorem is actually the calculation

$$\text{Index}(D) = h_E(t) - h_F(t),$$

(5.1)
where $E$ and $F$ denote two vector bundles over a differential manifold $M$, and

$$h(t) = \sum_{\lambda_n} e^{-t\lambda_n} = \text{Tr} e^{-t\Delta},$$

(5.2)

where $\Delta$ is the Laplacian operator on the vector bundle $E$ or $F$. $h(t)$ is called the heat function, which satisfies the heat equation

$$-\frac{\partial}{\partial t} h = \Delta h.$$  

(5.3)

Hence, the calculation on the index becomes the problem of solving the heat equation (5.3). One conveniently uses the Green function method to solve the equation, which gives

$$h(t) = \int_M K(t, x, x) \sqrt{g} d^n x, \quad x = (x^1, \cdots, x^n),$$

(5.4)

where $K(t, x, y)$ is determined by

$$\left( \frac{\partial}{\partial t} + \Delta_x \right) K(t, x, y) = 0, \quad K(0, x, y) = \delta(x - y).$$

(5.5)

For the Laplacian operator in Euclidean space, $\Delta_0 = \partial^2 = \partial_{\mu} \partial^\mu$, one can easily find

$$K_0(t, x, y) = \frac{1}{(4\pi t)^{n/2}} \exp \left[ -\frac{(x - y)^2}{4t} \right].$$

(5.6)

But for a Laplacian operator defined on a general vector bundle, it is impossible to solve (5.5) exactly. One can only find the asymptotic expansion form of $K(t, x, y)$ at $t \to 0^+$ [5],

$$K(t, x, y) = \sum_{i=0}^\infty t^{(i-n)/2} f_i(x) \sqrt{g}.$$  

(5.7)

Substituting (5.7) into the heat equation, we can obtain a recursive formula for $f_i(x)$. From Eqs. (5.1), (5.4), and (5.7), one can find that as $t \to 0^+$, the pole and regular terms (i.e., $i \neq n$) cancel, and hence

$$\text{Index}(D) = \text{Tr} \int_M \left[ f^E_n(x) - f^F_n(x) \right] \sqrt{g} d^n x.$$  

(5.8)

Eq. (5.8) shows that $f^E_n(x) - f^F_n(x)$ plays the role of characteristic classes.

In fact, the asymptotic expansion at $t \to 0^+$ is very complicated. It is just the large numbers of cancelations between the $i \neq n$ terms that make it possible to consider only the $i = n$ terms (see Eq. (5.8)). It is believed that the cancelations should be caused by a hidden supersymmetry-like algebraic structure in the heat equation approach. Hence, the following conjecture has been proposed.

**Theorem:** Let $f(t)$ and $g(t)$ be two solutions to a system of heat equations

$$-\frac{\partial}{\partial t} f = \Delta f, \quad \text{and} \quad -\frac{\partial}{\partial t} g = \Delta^\dagger g,$$

where $\Delta = D^\dagger D$ is an non-negative elliptic operator. Then there exists a graded algebra structure that determines solutions.
Appendix A

Nöther Theorem in a Mechanical System

The Nöther theorem establishes a direct relation between symmetry and the conservative law in a classical physical system. If the classical action is invariant under any transformations characterized by time-independent parameters, then the classical mechanical system has one conservative (i.e. time-independent) physical quantity corresponding to each parameter. Because successive transformations can form a Lie group, the Nöther theorem provides a way to analyze the algebraic features of physical systems. In quantum theory, the conservative physical quantities correspond to the generators of a Lie group.

Let us consider a classical mechanical system described by a Lagrangian $L[\phi_\mu(t), \dot{\phi}_\mu(t)]$. The classical action is

$$S = \int dt L[\phi_\mu(t), \dot{\phi}_\mu(t)], \quad \mu = 1, 2, \ldots, n,$$

where $\phi_\mu$ can be either commutative variables or Grassmann variables.

Assume that the action is invariant under certain transformations whose infinitesimal version is

$$\delta \phi_\mu = \epsilon_i f_{i\mu}(\phi), \quad i = 1, 2, \ldots, p$$

where $\epsilon_i$ are infinitesimal constant transformation parameters. Then we have

$$\delta S = \int dt \delta L = \int dt \left( \frac{\partial L}{\partial \phi_\mu} \delta \phi_\mu + \frac{\partial L}{\partial \dot{\phi}_\mu} \delta \dot{\phi}_\mu \right)$$

$$= \int dt \left[ \delta \phi_\mu \left( \frac{\partial L}{\partial \phi_\mu} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}_\mu} \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}_\mu} \delta \phi_\mu \right) \right]$$

$$= \int dt \epsilon_i \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{\phi}_\mu} f_{i\mu}(\phi) \right] = 0,$$

where we have used the Euler-Lagrange equation (3.4). In addition, when $\phi_\mu$ are Grassmann variables (see Appendix B), the anticommutative property must be taken into account in the calculation. Because $\epsilon_i$ are arbitrary parameters, Eq. (A.3) yields conservative laws

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{\phi}_\mu} f_{i\mu}(\phi) \right] = 0,$$
and we obtain $p$ conservative (or time-independent) quantities corresponding to $\epsilon_i$,

$$Q_i = \frac{\partial L}{\partial f_{i\mu}} f_{i\mu}(\phi) = \pi_\mu f_{i\mu}(\phi), \quad (A.5)$$

where $\pi_\mu$ is the conjugate momentum of $\phi_\mu$ (see Eq. (3.6)).

By Poisson brackets (3.13), the symmetry transformations (A.2) can be expressed as

$$\delta \phi_\mu = - [\epsilon_i Q_i, \phi_\mu]_{\text{PB}}. \quad (A.6)$$

At a quantum level, the variable becomes an operator, the Poisson bracket is replaced by a Lie bracket, and (A.6) becomes

$$\delta \hat{\phi}_\mu = - i [\epsilon_i \hat{Q}_i, \hat{\phi}_\mu] . \quad (A.7)$$

The finite version of Eq. (A.7) is

$$\hat{\phi}_\mu \rightarrow \exp \left[ -i \epsilon_i \hat{Q}_i \right] \hat{\phi}_\mu \exp \left[ i \epsilon_i \hat{Q}_i \right]. \quad (A.8)$$

Therefore, the successive transformations form a continuous Lie group since $\exp \left[ i \epsilon_i \hat{Q}_i \right]$ is a representation of a group element of the Lie group with generators $\hat{Q}_i$. 

Appendix B

Grassmann Variable Calculus

Grassmann variables are anticommutative functions, and they are essential dynamical variables in a theory with supersymmetry. The following introduces the calculus on Grassmann variables, and shows distinct features from usual commutative variables.

1. Grassmann algebra

Grassmann variables are defined by the algebra they satisfy. A Grassmann algebra \( \mathcal{A} \) is an algebra over the field \( \mathbb{R} \) or \( \mathbb{C} \) constructed from a set of generators \( \theta_i \) under anticommutative product [21]:

\[
\{ \theta_i, \theta_j \} \equiv \theta_i \theta_j + \theta_j \theta_i = 0, \quad i, j = 1, \ldots, n. \tag{B.1}
\]

Obviously, Eq. (B.1) implies \( \theta_i^2 = 0 \). It is easy to see that \( \mathcal{A} \) is a vector space of dimension \( 2^n \) on \( \mathbb{R} \) or \( \mathbb{C} \), with a basis of monomials of degree \( k \),

\[
\theta_{i_1} \cdots \theta_{i_k}, \quad 0 \leq k \leq n. \tag{B.2}
\]

2. Grassmann Parity

Grassmann parity is a useful notion for defining differentiation and integration over Grassmann variables. On the algebra \( \mathcal{A} \), there exists a simple automorphism \( P \) which is a reflection defined by

\[
P(\theta_i) = -\theta_i. \tag{B.3}
\]

Then under the action of \( P \) the basis listed in (B.2) behaves as,

\[
P(\theta_{i_1} \cdots \theta_{i_k}) = (-1)^k \theta_{i_1} \cdots \theta_{i_k}. \tag{B.4}
\]

3. Grassmann differentiation
The differentiation operator $D$ on $\mathcal{A}$ is a linear mapping $D : \mathcal{A} \to \mathcal{A}$ such that
\[ D(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 D(f_1) + \lambda_2 D(f_2), \] (B.5)
for $\lambda_1, \lambda_2 \in \mathbb{R}$ or $\mathbb{C}$ and $f_1, f_2 \in \mathcal{A}$, satisfying the condition
\[ D(f_1 f_2) = D(f_1) f_2 + P(f_1) D(f_2). \] (B.6)

$D$ can be explicitly represented by its action on the generators $\theta_i$ [21],
\[ \frac{\partial}{\partial \theta_i} \theta_j = \delta_{ij}. \] (B.7)

The differential operators $\partial/\partial \theta_i$, as a representation of $D$ realized on $\mathcal{A}$, can be understood as derivatives with respect to $\theta_i$ as the case of commutative variables. It is easy to find $\partial/\partial \theta_i$, together with $\theta_i$ considered as operators acting on $\mathcal{A}$ by left-multiplication, satisfying the following anticommutation relations [21],
\[ \theta_i \theta_j + \theta_j \theta_i = 0, \quad \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial \theta_j} \frac{\partial}{\partial \theta_i} = 0, \quad \text{and} \quad \theta_i \frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial \theta_j} \theta_i = \delta_{ij}. \] (B.8)

Thus these differential operators $\partial/\partial \theta_i$, and generators $\theta_i$, form a Clifford algebra (see Appendix C).

4. Grassmann Integration

The integration operator $I$ is a linear operator acting on $\mathcal{A}$ such that
\[ I(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 I(f_1) + \lambda_2 I(f_2) \] (B.9)
satisfying
\[ ID = 0, \quad DI = 0, \] (B.10)
and
\[ I(fg) = I(f)g \quad \text{if} \quad D(g) = 0. \] (B.11)

The first condition in Eq. (B.10) means that the integration of a total derivative term should vanish in the absence of boundary terms. The second condition implies that once a Grassmann variable has been integrated over, the value obtained by the integration no longer depends on this variable. Finally the condition in Eq. (B.11) implies that a factor whose derivative with respect to a Grassmann variable vanishes can be taken out of the integration.

The explicit representation on the generators $\theta_i$ of the integral operator $I$ that satisfies the above three conditions, denoted as $\int d\theta_i$, can be chosen to be $\partial/\partial \theta_i$:
\[ \int d\theta_i f = \frac{\partial}{\partial \theta_i} f, \quad \forall f \in \mathcal{A}. \] (B.12)

This is one of the most distinct features of Grassmann integration from the integration over commutative variables.
5. Change of Variables in A Grassmann Integral

We have the following result on the change of variable in multiple variable integrations over Grassmann variables:

**Theorem B.1.** Consider the following multiple variable integration over a number of Grassmann variables,

\[ \int d\theta_1 \cdots d\theta_n f(\theta). \]  \hspace{1cm} (B.13)

If there is a change of variables,

\[ \theta_i = \theta_i(\theta'), \]  \hspace{1cm} (B.14)

then there exists

\[ \int d\theta_1 \cdots d\theta_n f(\theta) = \int d\theta'_1 \cdots d\theta'_n J(\theta') f(\theta'), \]  \hspace{1cm} (B.15)

with the Jacobian determinant,

\[ J = \det \left( \frac{\partial \theta_i}{\partial \theta'_j} \right). \]  \hspace{1cm} (B.16)

Use the following simple example to verify the result. Making the change of variables by a linear transformation constructed from commutative numbers,

\[ \theta_i = a_{ij} \theta'_j, \]  \hspace{1cm} (B.17)

and using the identity

\[ \theta_n \cdots \theta_1 = a_{ni} \theta'_n \cdots a_{i1} \theta'_1 = a_{ni} \cdots a_{i1} \epsilon_{in} \cdots i_1 \theta'_n \cdots \theta'_1 = \det(a_{ij}) \theta'_n \cdots \theta'_1, \]  \hspace{1cm} (B.18)

and substituting (B.18) into the following integration

\[ 1 = \int d\theta_1 \cdots d\theta_n f(\theta), \]  \hspace{1cm} (B.19)

we obtain

\[ J = \det(a_{ij}) = \det(\partial \theta / \partial \theta'). \]

6. Gaussian Integrals with Grassmann Variables

1. Gaussian integral over a pair of complex conjugated Grassmann numbers

To introduce Gaussian integrals over Grassmann variables, we need to understand the Gaussian integral over a pair of complex conjugated Grassmann numbers.

First, given two real Grassmann numbers \( \theta_1 \) and \( \theta_2 \), define two conjugate complex Grassmann numbers,

\[ \eta = \frac{1}{\sqrt{2}} (\theta_1 + i\theta_2), \quad \text{and} \quad \bar{\eta} = \frac{1}{\sqrt{2}} (\theta_1 - i\theta_2). \]  \hspace{1cm} (B.20)
It is easy to verify
\[ \eta^2 = \bar{\eta}^2 = 0, \quad \text{and} \quad \eta \bar{\eta} + \bar{\eta} \eta = 0. \tag{B.21} \]
Then with the definition
\[ d\eta = \frac{1}{\sqrt{2}} (d\theta_1 - i d\theta_2), \quad \text{and} \quad d\bar{\eta} = \frac{1}{\sqrt{2}} (d\theta_1 + i d\theta_2), \tag{B.22} \]
we obtain
\[ \int d\eta \eta = \int d\bar{\eta} \bar{\eta} = 1, \quad \int d\eta \bar{\eta} = \int d\bar{\eta} \eta = 0, \quad \text{and} \quad \int d\eta = \int d\bar{\eta} = 0. \tag{B.23} \]
Then we have
\[ \int d\eta d\bar{\eta} \exp (-\eta \bar{\eta}) = \int d\eta d\bar{\eta} (1 - \eta \bar{\eta}) = \int d\eta d\bar{\eta} = 1. \tag{B.24} \]

2. Gaussian integral over \( n \) pairs of complex conjugated Grassmann numbers
Let us consider a Gaussian integral over \( n \) pairs of complex conjugate Grassmann numbers:
\[ I(a) = \int d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2 \cdots d\bar{\eta}_n d\eta_n \exp (-\bar{\eta}_i a_{ij} \eta_j). \tag{B.25} \]
Using the change of variables
\[ a_{ij} \eta_j = \eta_i', \tag{B.26} \]
and the properties shown in (B.15) and (B.16),
\[ d\eta_1 \cdots d\eta_n = (\det a) \ d\eta_1' \cdots d\eta_n', \quad a = (a_{ij}), \tag{B.27} \]
we have
\[ I(a) = \det a \ \int d\bar{\eta}_1 d\eta_1' \cdots \int d\bar{\eta}_n d\eta_n' \exp (-\bar{\eta}_i a_{ij} \eta_j) = \det a. \tag{B.28} \]
Further, using the result (B.28), (B.20) and (B.22) we obtain the Gaussian integral over \( n \) real Grassmann variables,
\[ \int d\theta_n \cdots d\theta_1 \exp \left( -\frac{1}{2} \theta_i a_{ij} \theta_j \right) = \begin{cases} \pm \sqrt{\det a}, & n=\text{even positive integer;} \\ 0, & n=\text{odd positive integer}. \end{cases} \tag{B.29} \]
Appendix C

Clifford Algebra and Spinor

Spinors are elements of a complex vector space, and are represented by complex row or column vectors. They furnish a representation for the Clifford algebra.

1. Clifford Algebra and Matrix Representation

A Clifford algebra with \( n \) generators \( \gamma_\mu, \mu = 1, \ldots, n \) is defined by the following relations,

\[
\{ \gamma_\mu, \gamma_\nu \} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu \nu}, \quad \mu, \nu = 1, 2, \ldots, n.
\] (C.1)

By defining

\[
\gamma_{\mu_1 \mu_2 \cdots \mu_p} = \frac{1}{p!} \gamma_{[\mu_1} \gamma_{\mu_2} \cdots \gamma_{\mu_p]},
\] (C.2)

we can construct a finite group \( C_n \) under multiplication,

\[
C_n = \{ \pm 1, \pm \gamma_1, \pm \gamma_{\mu_1 \mu_2}, \cdots, \pm \gamma_{\mu_1 \cdots \mu_n} \}.
\] (C.3)

Because \( \gamma_{\mu_1 \mu_2 \cdots \mu_p} \) is completely antisymmetric with respect to the indices \( \mu_1, \mu_2, \cdots, \mu_p \), there are

\[
\binom{n}{p}
\]
distinct \( \gamma_{\mu_1 \mu_2 \cdots \mu_p} \)'s. Hence the order of \( C_n \) is

\[
|C_n| = 2 \sum_{p=0}^{n} \binom{n}{p} = 2 \times 2^n = 2^{n+1}.
\] (C.4)

In the following we work out the matrix representation of the Clifford algebra in the case that \( n \) is even, i.e., \( n = 2k \), since it is the only case we need in this paper. First, consider the following list of results in the theory of a finite group [29]:
1. **Theorem C.1.** The conjugation class \([a]\) of a group element \(a \in G\) is defined by

\[
[a] = \{gag^{-1}, \forall g \in G\}.
\] (C.5)

Then the number of irreducible representations of any finite group \(G\) equals the number of its conjugation class.

2. **Theorem C.2.** The commutator group \(C(G)\) of a group \(G\) is defined by

\[
C(G) = \{aba^{-1}b^{-1}, \forall a, b \in G\}.
\] (C.6)

Let \(N\) denote the number of inequivalent one-dimensional representations of a finite group \(G\). Then

\[
N = \frac{|G|}{|C(G)|},
\] (C.7)

where \(|G|\) and \(|C(G)|\) are orders of \(G\) and \(C(G)\).

3. **Theorem C.3.** Let \(|G|\) denote the order of a finite group \(G\). If \(G\) has \(p\) irreducible inequivalent representations of dimension \(d_i, i = 1, \cdots, p\), then

\[
|G| = \sum_{i=1}^{p} (d_i)^2.
\] (C.8)

The following applies the above three results to \(C_n\). First, by using Eq. (C.1) it is easy to show that \(C_n\) has \(2^n + 1\) conjugation classes:

\[
[+1], [-1], [\gamma], [\gamma_{\mu_1}], \cdots, [\gamma_{\mu_1\mu_2}\cdots\mu_n].
\] (C.9)

Hence by Theorem C.1, \(C_n\) has \(2^n + 1\) irreducible representations.

Second, for \(n = 2k\), we have

\[
C(C_n) = \{+1, -1\},
\] (C.10)

and hence the number of inequivalent one-dimensional irreducible representations is

\[
N = \frac{|C_n|}{|C(C_n)|} = \frac{2^{n+1}}{2} = 2^n.
\] (C.11)

From above, the total number of irreducible representations is \(2^n + 1\), leaving only one irreducible representation whose dimension is greater than 1.

According to Theorem C.3., we have

\[
2^{n+1} = 1^2 2^n + d^2,
\] (C.12)

where \(d\) denotes the dimension of the only irreducible representation whose dimension is greater than 1. Hence

\[
d = 2^{n/2}.
\] (C.13)
Therefore, $\gamma_\mu$ can be represented by $2^{n/2} \times 2^{n/2}$ matrices. Consequently, a spinor $\psi$ is column vector with $2^{n/2}$ complex components,

$$
\psi = [\psi_\alpha] = \begin{bmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_{2^{n/2}} 
\end{bmatrix}.
$$

The space formed by spinor provides a matrix representation space for $\gamma_\mu$.

2. Decomposition of Spinor Space $S$

When $n = 2k$, define a matrix from the $n$ $\gamma$-matrices,

$$
\gamma_{n+1} = \gamma_1 \gamma_2 \cdots \gamma_n.
$$

A straightforward calculation shows

$$
\gamma_{n+1}^2 = (-1)^{n(n-1)/2}I,
$$

where $I$ is an $2^{n/2} \times 2^{n/2}$ identity matrix. This means

$$
\begin{align*}
\gamma_{n+1}^2 &= I \text{ for } n = 0 \text{ mod } 4; \\
\gamma_{n+1}^2 &= -I \text{ for } n = 2 \text{ mod } 4.
\end{align*}
$$

Subsequently, for $n = 2 \text{ mod } 4$, we define

$$
\psi_\pm = \frac{1}{2}(I \pm \gamma_{n+1})\psi, \quad \text{and} \quad \gamma_{n+1}\psi_\pm = \pm\psi_\pm.
$$

For $n = 4 \text{ mod } 4$, we define

$$
\begin{align*}
\tilde{\gamma}_{n+1} &= i\gamma_{n+1}, \\
\psi_\pm &= \frac{1}{2}(I \pm \tilde{\gamma}_{n+1})\psi, \quad \tilde{\gamma}_{n+1}\psi_\pm = \pm\psi_\pm.
\end{align*}
$$

$\psi_\pm$ are called chiral (or Weyl) spinors with chirality $\pm 1$. Therefore, the spinor space can decompose a direct sum of two subspaces composed of chiral spinors as,

$$
S = S^+ \oplus S^-,
$$

$$
S^+ = \{\psi_+\}, \quad S^- = \{\psi_-\}.
$$
Bibliography


